ANALYSIS OF TELEGRAPH EQUATION AND ITS SOLUTION USING SPECTRAL COLLOCATION METHOD WITH DIFFERENT BASIS FUNCTIONS

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Abstract: In this paper, the celebrated telegraph equation which is an one dimensional hyperbolic PDE is studied using spectral collocation method. Three sets of special functions such as Legendre, Chebyshev and Jacobi functions are considered for the present study and their role as a basis function in obtaining a more accurate solution with less computational effort is analysed. The change in the parametric values of the Inductance and Resistance affecting the solution of the telegraph equation is also computed which has not been attempted by any other researchers. The novelty of the present method lies in obtaining very good accuracy at less number of collocation points. Several available results in the latest literature are compared to show the efficiency of the present method of solution. Several tables and figures are presented to support our analysis.

Keywords: Chebyshev, Legendre and Jacobi polynomials, Spectral Collocation Method, Telegraph Equation.

Introduction: As is popularly known, several physical and engineering processes when modelled mathematically results in partial differential equations. Thus, a search for a more accurate solution of these differential equations using the latest developed methods and techniques is a popular research study in the literature. One of the famous Partial differential equations that models the Telegraph equation is a study of electric voltage and the current in a double conductor with distance x and time t. Telegraph equation has been attracted by researchers in past few decades due to its application in the vibration of structures, signal analysis, wave propagation, microwaves and radio frequency fields and random walk theory. In recent years, many numerical schemes were developed in order to study this equation [1, 2, 3,6].

The Present Study: The telegraph equation considered here is a one-dimensional second order hyperbolic PDE givenby [3]:

$$c^2V_{xx} = V_{tt} + (\alpha+\beta)V_t + \alpha\beta V \qquad (1)$$

Where, $c^2 = \frac{1}{LC}$, $\alpha = \frac{G}{L}$ and $\beta = \frac{R}{C}$. Here, $V(x,t)$ is the voltage at any point x and any time t, on the cable. R, C, L, G denotes the resistance of the cable, the capacitance to the ground, inductance of the cable and conductance to the ground respectively. In this paper, we have studied (1) using spectral collocation method with Legendre, Chebyshev and Jacobi functions as three different basis functions.

Method of Solution: Spectral methods are a class of mathematical techniques to numerically solve certain differential equations [4,5]. In this section, we apply spectral collocation for the telegraph equation. The spectral solution for the equation (1) in the interval $[a,b]$ can be written as

$$V(x,t) \approx V^N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\phi_i((2x-(b-a))/(b+a))\phi_j((t-(b-a))/(b+a)) \qquad (2)$$

We will represent $\phi_i((2x-(b-a))/(b+a))$ as $\phi_i(x)$. Here, N denotes the number of collocation points. The different special functions considered are represented as $T_n((2x-(b-a))/(b+a))$ for

Chebyshev polynomials, $L_n((2x-(b-a))/(b+a))$ for Legendre Polynomial and $P_n^{1,1}((2x-(b-a))/(b+a))$ for Jacobi Polynomials. The essence of the present scheme is that we obtain its residual function and force it to go to zero at certain sets of collocation points. The residual function is obtained on substituting the spectral solution in (1) as given by:

$$R(x,t) = V_{tt}^N + (\alpha+\beta)V_t^N + (\alpha\beta)V^N - c^2 V_{xx}^N \qquad (3)$$

where,

$$V_{tt} = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\phi_i(x)\frac{d^2\phi_j(t)}{dt^2}, V_t = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\phi_i(x)\frac{d\phi_j(t)}{dt}, V_{xx} = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\frac{d^2\phi_i(x)}{dx^2}\phi_j(t)$$

The set of collocation points considered for Chebyshev polynomial as basis function are $\mathbf{x_i} = \frac{1}{2}(\mathbf{a}+\mathbf{b}-(\mathbf{b}-\mathbf{a})\mathbf{cos}(\pi * \mathbf{i})/\mathbf{N})$, where i varies from 1 to $N-1$ and for Legendre and Jacobi polynomial as basis functions, we consider the zeros of their first derivatives as the collocation points. Substituting the appropriate collocation points in (3), we get a system of $(N+1)^2$ algebraic equations with $(N+1)^2$ unknowns. On solving these algebraic equations, we get the unknown coefficients a_{ij}. These computed values of a_{ij} are replaced in (2) to obtain spectral solution.

Numerical Results: As mentioned earlier, the telegraph equation (1) is analysed using spectral collocation method with three different basis functions. We have used Mathematica 8.0 for all our Computations. It was noted that the maximum CPU time taken for any of computation was utmost 5.314 seconds for all the results computed here.The accuracy of the present method is analysed using absolute error for a given t_j and x_i by

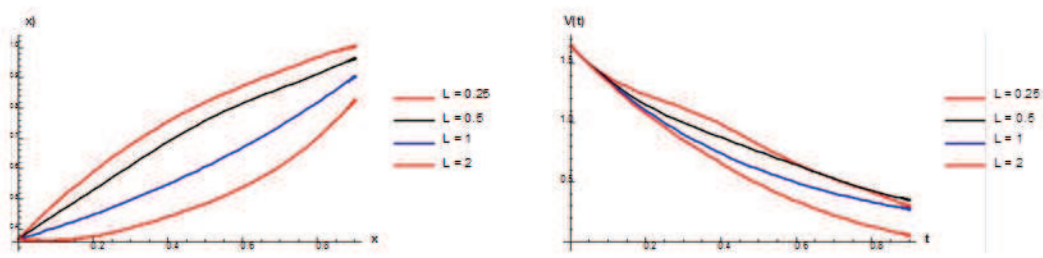
$$\text{AbsoluteError} := |V(x_i,t_j) - V^N(x_i,t_j)|, \text{ Maximum Absolute Error} := \max_{i,j}|V(x_i,t_j) - V^N(x_i,t_j)| \quad (5)$$

where $V(x_i,t_j)$ and $V^N(x_i,t_j)$ are the exact and spectral solution at various interior collocation points x_i and for a given t_j.We present a sample of two representative results in which the first example considers a specific set of initial and boundary conditions that are available in the literature to compare the accuracy of results and the number of collocation points. In the second example we compute the results for changing the values of L and R.

Example 4.1 *In this example, we consider the telegraph equation (1) with initial and boundary conditions as follows [6]:*

$$\text{i.c.s } V(x,0) = e^x, \quad V_t(x,0) = 2e^x, \quad \text{b.c.s} V(0,t) = e^{-2t}, \quad V(1,t) = e^{1-2t} \qquad (6)$$

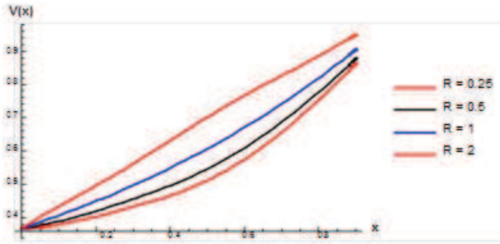
Fig. 1 represents the spectral solution of $V(x,t)$ versus x for different values of L for a fixed value of $R=1$ at certain time $t=1$. From Fig 1 (a) and (b), It is observed that the Voltage decreases as inductance in the cable increases. From Fig. 1 (c) and (d) it is clear that Voltage is directly proportional to R. By several computations, we also observe that C shows that same behaviour as that of inductance L and G shows a similar behaviour as that of R, with respect to voltage.



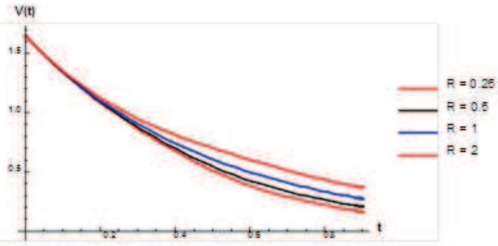
(a) Solution for different values of L for $R=1$ and $t=1$

(b) Solution for different values of L for $R=1$ and $x=1$

Fig. 1: Effect of parameters L, R with C=1 and G=1 for (1)



(c) Solution for different values of R for $L=1$ and $t=1$



(d) Solution for different values of R for $L=1$ and $x=1$

Example 4.2: In this example, we consider the non-homogeneous telegraph equation with initial and boundary conditions given as follows [1,2] :

$$V_{tt} + V_t + V = V_{xx} + g(x,t) \quad (7)$$

with i.c.s $\boldsymbol{V(x,0) = x^2, V_t(x,0) = 1}$ **and b.c.s** $\boldsymbol{V(0,t) = t, V(1,t) = 1 + t}$. The exact solution of the above example is given by $V(x,t) = e^{x-2t}$. Table 1 displays the comparison of absolute error with Bessel Collocation method (BCM) for $t = 1$ at different spacial values x. Comparison of Maximum absolute error obtained using the present method with results to Galerkin method (GM) and Bessel Collocation method (BCM) for different Values of N is tabulated in Table 2. From these table, we note that $@x = 0.375$, for BCM absolute error is of order 10^{-7}, whereas for present method absolute error is of order atmost 10^{-9}.

Table 1: Comparison of Absolute Error for (7) with BCM [2]

x	BCM [2]	Present Results at $N = 7$		
	$N = 7$	Chebyshev	Legendre	Jacobi
0.125	1.25×10^{-7}	2.84008×10^{-9}	3.81697×10^{-9}	1.09736×10^{-9}
0.250	1.06×10^{-8}	5.03701×10^{-9}	6.99714×10^{-9}	2.44904×10^{-10}
0.375	3.72×10^{-7}	6.59076×10^{-9}	9.07020×10^{-9}	4.93711×10^{-10}
0.500	6.21×10^{-7}	7.16478×10^{-9}	9.78706×10^{-9}	3.43189×10^{-10}

Table 2: Comparison of Maximum Absolute Error for (7) with BCM [2] and GM [4] for Different Values of N

N	GM [4]	BCM [2]	Present Results		
			Chebyshev	Legendre	Jacobi
3	3.317×10^{-3}	–	2.510×10^{-3}	3.028×10^{-3}	3.688×10^{-3}
5	1.574×10^{-5}	3.9×10^{-4}	1.725×10^{-5}	3.174×10^{-5}	6.263×10^{-5}
7	3.482×10^{-7}	1.4×10^{-6}	5.824×10^{-8}	1.183×10^{-7}	2.738×10^{-7}

Fig. 2 represents Contour plot of absolute error for example 4.2 with respect to different basis functions. It is observed that the area of the core region is least in the figure (a) compared to figures (b) and (c). Thus, it can be said that, the use of Chebyshev polynomial as basis functions give a better accuracy compared to the Legendre and Jacobi polynomials.

Conclusion: We have analysed Telegraph equation using spectral collocation method with three different basis functions. All our computations and results are presented in the form of tables or as figures. From Fig 1 we note that Voltage $V(x,t)$ is directly proportional to R and G and inversely proportional to L and C. The changes in the solution by varying parameters L, C, R and G is captured in this paper. The novelty of the method lies in obtaining very good accuracy at low number of collocation points which can be seen from Table 1 and 2. Fig. 2 shows Chebyshev polynomial as basis function gives good accuracy compared to other two polynomials as basis functions.

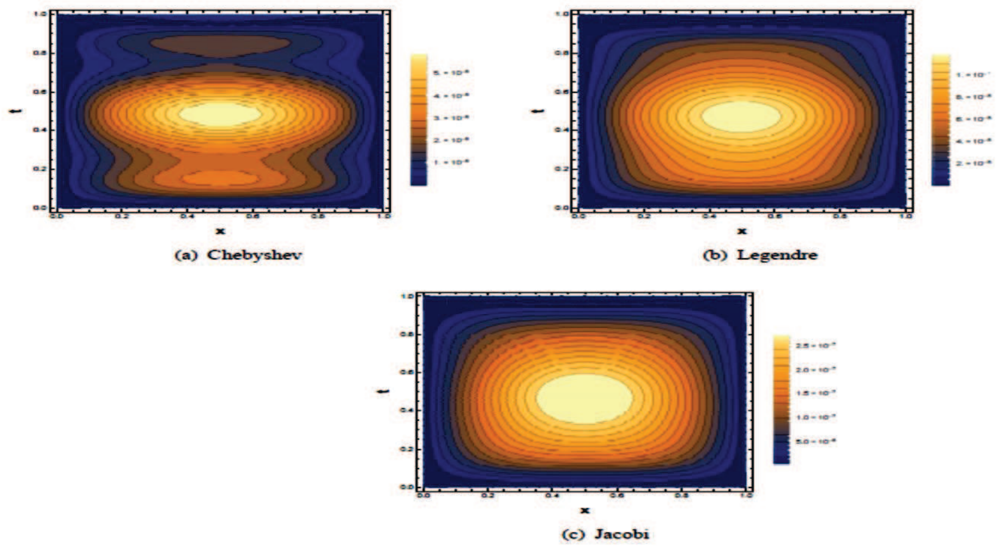


Fig. 2 Contour Plot for absolute error for Example 4.2 with respect to three different basis functions

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