DOMINATOR CHROMATIC NUMBER OF SOME GRAPHS

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Abstract: A dominator colouring of a graph G is a proper colouring of G in which each vertex dominates every vertex of at least one colour class. The dominator chromatic numbers have been determined for many families of graphs like Cycle , Path , Star graph , Prime graph, Wheel graph etc . In this paper the dominator chromatic number of Complete binary trees, Book graphs and Triangular snakes have been determined .

Keywords: Dominator colouring , Dominator chromatic number, Complete binary tree, Book graphs, Triangular Snakes.

1. Introduction: Graph colouring and domination are two important branches of Graph Theory . Colouring vertices and edges of a graph is an easy job but when the number of colours is restricted it becomes a tough task . Similar problem is there for domination. A dominator colouring of a graph G is a proper colouring of G in which each vertex dominates every vertex of at least one colour class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G [2]. The concept of dominator colourings was introduced by Gera, R.S Horton. and C. Rasmusssen, in 2006 [1].

Determining dominator chromatic number of a graph is a difficult problem. It seems to be challenging while dealing with families of graphs. Many researchers have made attempt in this line . Till now dominator chromatic numbers have been determined for nearly 35 families of graphs. This paper adds four more families to this collection.

2. Complete Binary Tree

Definition: [4]: A Complete binary tree is a binary tree in which all interior nodes have two children and all leaves have the same depth (or) same level.

2.1 Theorem: Let B_K be the complete binary tree with k levels with $K \geq 4$. Then

$$\chi_d(B_k) = \begin{cases} \frac{4}{7}\,[\,8^{l} - 1] + 3, & \text{if } k = 3l \\ \frac{1}{7}\,[\,8^{l+1} - 1] + 2, & \text{if } k = 3l + 1 \\ \frac{2}{7}\,[\,8^{l+1} - 1] + 2, & \text{if } k = 3l + 2 \end{cases}$$

Proof: Let $V(B_k) = \cup_{i=0}^{k} V_i$, where $V_i = \{v_i^1, v_i^2, \ldots, v_i^{2^i}\}$, i=0,1,2,...k` and edge set $E(B_k) = \cup_{i=0}^{k} E_i$, where $E_i = \left\{ v_i^1 v_{i+1}^1,\ v_i^1 v_{i+1}^2,\ v_i^2 v_{i+1}^3, v_i^2 v_{i+1}^4, \ldots, v_i^{2i} v_{i+1}^{2^{i+1}-1}, v_i^{2^i} v_{i+1}^{2^{i+1}} \right\}$

The vertices of V_i form the i^{th} level of B_k.

$|V(B_k)| = 2^k + 2^{k-1} + \ldots + 2 + = \frac{2^{k+1}-1}{2-1} = 2^{k+1} - 1$

$\Delta(B_k) = 3$

Therefore, $\gamma(B_k) \geq \frac{1}{4}|V(B_k)| = \frac{1}{4}[2^{k+1}-1]$

Case 1: Let k = 3l

Let S = $V_{K-1} \cup V_{K-1} \cup ,\ldots, \cup V_5 \cup V_2 \cup V_0$

$|S| = 2^{3l-1}+2^{3l-4}+\ldots+2^5+2^2+2^0 = 2^2[2^{3l-3}+2^{3l-6}+2^3+1]+1$

$|S| = 2^2[8^{l-1}+8^{l-2}+\ldots+8+1]+1 = 2^2\left[\frac{8^l-1}{8-1}\right]+1$

$|S| = \frac{4}{7}[8^l-1]+1$

S is a minimal dominating set

Therefore, $\gamma(B_k) = \frac{4}{7}[8^{l-1}]+1$

Assign separate colours to each vertex in minimal dominating set S. This needs γ colours.

Assign(γ+1)th colours to the vertices in the levels k, k-3, k-6,..., 3

Assign (γ+2)th colours to the vertices in the levels k-2, k-5,....,4,1

This colouring is a dominator colouring

$\gamma(B_{3l})+1 \leq \chi_d(B_{3l}) \leq \gamma(B_{3l})+2$, since B_{3l} is a tree.

S is an independent dominating set.

No two vertices of S can be in the same colour class of any dominator colouring .(suppose x,y ϵ S and x,y ϵ V_i for some i, $1 \leq I \leq \gamma$. Then x does not dominate any colour class)

$B_{3l} \setminus S$ is not independent and $\chi(B_{3l} \setminus S) \geq 2$

Hence $\chi_d(B_{3l}) = \gamma+2 = \frac{4}{7}[8^l-1]+3$

Case 2: Let K =3l+1

Let $S = V_{K-1} \cup V_{K-3} \cup \ldots \cup V_1 \cup V_3 \cup V_0$

$|S| = \frac{8^{l+1}-1}{8-1}$;$|S| = \frac{1}{7}[8^{l+1}-1]$,

S is a minimal dominating set ,Therefore, $\gamma(B_k) = \frac{1}{7}[8^{l+1}-1]$

Assign separate colours to each vertex in minimal dominating set S. This needs γ colours.

Assign (γ+1)th colours to the vertices in the levels k-2,k-4,...,1

Assign (γ+2)th colours to the vertices in the levels k,k-5,....,2

This colouring is a dominator colouring. $\chi_d(B_{3l+1}) = \gamma(B_{3l+1})+2 = \frac{1}{7}[8^{l+1}-1]+2$.

Case 3: Let k=3l+2

Let $S = V_{K-1} \cup V_{K-3} \cup \ldots \cup V_4 \cup V_1$

$|S| = 2^{3l+1}+2^{3l-2}+2^{3l-5}+\ldots+2^4+2 = |S| = \frac{2}{7}[8^{l+1}-1]$,S is a minimal dominating set .

Therefore, $\gamma(B_k) = \frac{2}{7}[8^{l+1}-1]$

Assign separate colours to each vertex in minimal dominating set S. This needs γ colours.

Assign (γ+1)th colours to the vertices in the levels k-1,k-4,...,3

Assign (γ+2)th colours to the vertices in the levels k-2,k-5,....,4,1

This colouring is a dominator colouring. $\chi_d(B_{3l+2}) = \gamma(B_{3l+2})+2 = \frac{2}{7}[8^{l+1}-1]+2$

3. *Book Graphs:*

Definition: [3]: R_k ,Book with k rectangular pages is the graph with $V(R_k) = \{u_1, u_2, v_i^1, v_i^2 : 1 \leq i \leq k\}$ and $E(R_k) = \{u_1 u_2\} \cup \{u_1 v_i^1 : 1 \leq i \leq k\} \cup \{u_2 v_i^2 : 1 \leq i \leq k\}$

Definition: [3]: T_k the book with k triangular pages is defined as the graph with vertex set $V(T_k) = \{u_1, u_2, v_1, v_2 \ldots : v_k\}$ and edge set $E(T_k) = \{u_1u_2, u_1v_i, u_2v_i \ldots, 1 \leq i \leq k\}$

Theorem: 3.1: $\chi_d(R_k) = 4$ for k>1 and $\chi_d(R_1) = 2$

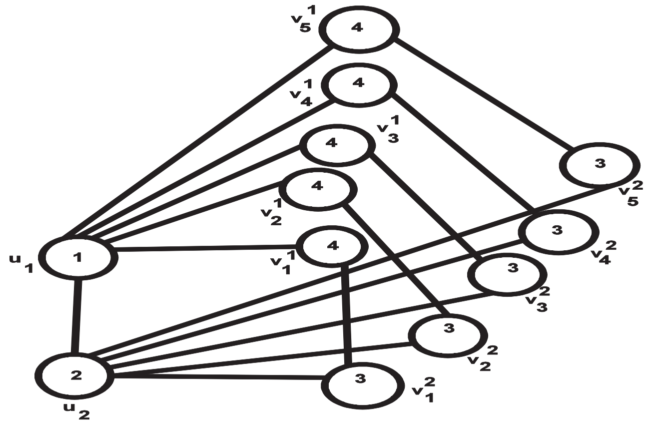
Proof: It is easy to check that , $\chi_d(R_1) = 2$

Let, k>1 $\chi_d(R_k) \geq \chi(_k) \geq 2$

If possible, let $\chi_d(R_k) = 2$

Let (V_1, V_2) be a dominator colouring
W.l.o.g ,let $u_1 \in V_1$ and $u_2 \in V_2$
v_1^1 is not adjacent to u_2. v_1^1 does not dominate V_2.
Therefore, v_1^1 dominates V_1.
Hence, $V_1 = \{u_1 , v_1^2\}$. V_2 contains v_2^1 and v_2^2
$\Rightarrow\Leftarrow$
Therefore, $\chi_d (R_k) > 2$. If possible, $\chi_d (R_k) = 3$
Let (V_1 ,V_2 ,V_3) be a dominator colouring
W.l.o.g , let $u_1 \in V_1$ and $u_2 \in V_2$
v_1^1 is not adjacent to u_2.v_1^1 does not dominate V_2. So, v_1^1 dominates either V_1 (or) V_3

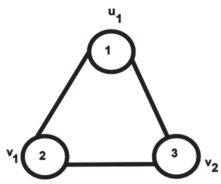
Case1: Suppose that v_1^1 dominates V_1
Then, $V_1 = \{u_1 , v_1^2\}$. $v_1^1 \in V_2$ (or) $v \in V_3$
Sub case:(i) Let $v_1^1 \in V_2$
v_2^1 is not adjacent to v_1^2
Therefore, v_1^1 does not dominate either V_1 (or) V_2 . v_2^1 should dominates V_3. Hence $V_3 = \{ v_2^1\}$ (or) $V_3 = \{ v_2^2\}$. If $V_3 = \{v_2^1\}$ then $V_2 = \{ u_2 , v_2^2\}$
Which is a contradiction.(Since u_2 is adjacent to v_2^2)
If $V_3 = \{ v_2^2 \}$ then V_2 contains u_2, v_1^1 , v_2^1, ..., v_1^2 does not dominate any colour class
$\Rightarrow\Leftarrow$
Sub Case: 2 Let $v_1^1 \in V_3$
$V_1 = \{ u_1 , v_1^2\}$, $u_2 \in V_2$ (or) $v_1^1 \in V_3$. v_2^1 is not adjacent to v_1^2 , u_2 and v_1^1.
Therefore, v_2^1 does not dominate any colour class.
$\Rightarrow\Leftarrow$
Case: 2 Suppose v_1^1 dominates V_3
Then $V_3 = \{ v_1^1\}$ and $V_1 = \{ u_1 , v_2^2\}$ and $V_2 = \{ u_2, v_2^1, v_1^2\}$
v_2^2 does not dominate any colour class. so, it is not a dominator colouring.
$\Rightarrow\Leftarrow$
Hence $\chi_d (R_k) > 3$
Describe a 4-colouring of B_k as follows,
$V_2 = \{ u_1\}$, $V_2 = \{ u_2 \}$, $V_3 = \{, v_1^1, v_2^1 , \ldots , v_k^1\}$, $V_4 = \{ v_1^2, v_2^2, \ldots , v_k^2\}$
u_1 dominates V_2, u_2 dominates V_2, v_i^1 dominates V_1 for $1 \le i \le k$, v_i^2 dominates V_2 for $1 \le i \le k$
Therefore, (V_1, V_2, V_3, V_4) is a dominator colouring.
Hence, $\chi_d (R_k) = 4$.



3.2 Theorem: $\chi_d(T_k) = 3$ for $k \geq 1$
Proof: T_k contains triangle.
Hence $\chi_d(T_k) \geq \chi(T_k) = 3$
consider the 3-colouring given below,
$V_1 = \{u_1\}$, $V_2 = \{u_2\}$, $V_3 = \{ v_1 v_2,\ldots,v_k\}$
It is a proper 3-colouring.
u_1 dominates V_1 , u_2 dominates V_2.....v_i dominates V_1 as well as V_2
The colouring given above is a dominator colouring. Therefore, $\chi_d(T_k) = 3$

4. Triangular Snakes
Definition: [5]: T_k , the triangular snakes with k triangular pages be defined as the graph with vertex set $V(T_k)=\{ u_1, u_2,\ldots, u_k, v_1, v_2,\ldots, v_k, v_{k+1}\}$ and edge set
$E(T_k)=\{v_i v_{i+1}:1 \leq i \leq k \} \cup \{ u_i v_i , u_i v_{i+1} , 1\leq i \leq k \}$.

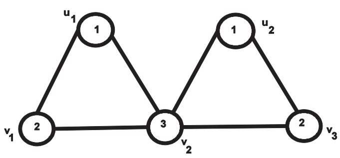
4.1 Theorem: $\chi_d(Tk) = \begin{cases} 3 & \text{for } k = 1,2 \\ 4 & \text{for } k = 3,4 \end{cases}$
Proof: ***Case 1:*** k=1



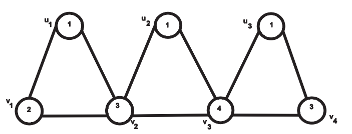
3-dominator colouring of T_1

$\chi_d(T_1) \geq \chi(T_1) = 3$ and the figure gives a dominator colouring with 3 colours.
Therefore $\chi_d(T_1) = 3$

Case 2: Let k=2



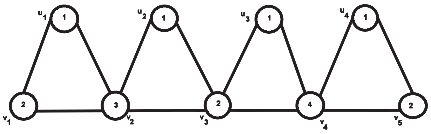
3-dominator colouring of T_2
$\chi_d(T_2) \geq \chi(T_2) = 3$ .The figure depicts a 3-dominator colouring of T_2. Therefore $\chi_d(T_2) = 3$
Case3: Let k=3



4-dominator colouring of T_3

T_3 contains triangle . At least 3 colors are needed to colour the vertices of T_3.
Therefore $\chi(T_3) \geq 3$ and $\chi_d(T_3) \geq 3$
If possible ,let (V_1, V_2, V_3) be a dominator colouring of T_3
No two of u_1, v_1, v_2 can be in the same colour.
W.l.o.g , let $u_1 \in V_1$, $v_1 \in V_2$, $v_2 \in V_3$
u_3 is not adjacent to u_1 and , v_1 and , v_2
Therefore, u_3 does not dominates any colour class.
Which is a contradiction
Therefore, $\chi_d(T_3) \geq 3$
Describe a 4-coloring for T_3 as follows,
$V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1\}$, $V_3 = \{v_2, v_3\}$, $V_4 = \{v_3\}$
clearly , it is a dominator coloring
Therefore, $\chi_d(T_3) = 4$.

Case 4: Let k =4



4-dominator colouring of T_4

T_4 contains triangle . At least 3 colors are needed to colour the vertices of T_4 properly.
Therefore $\chi_d(T_4) \geq 3$
If possible ,let (V_1, V_2, V_3) be a dominator colouring of T_4
No two of u_1, v_1, v_2 can be in the same colour.
W.l.o.g , let $u_1 \in V_1$, $v_1 \in V_2$, $v_2 \in V_3$
u_3 is not adjacent to u_1 and v_1 and v_2
Therefore, u_3 does not dominates any colour class.
Which is a contradiction
Therefore, $\chi_d(T_4) \geq 3$
Describe a 4-coloring for T_4 as follows,
$V_1 = \{u_1, u_2, u_3, u_4, \}$, $V_2 = \{v_1, v_3, v_5\}$, $V_3 = \{v_2\}$, $V_4 = \{v_4\}$
clearly , it is a dominator coloring
Therefore, $\chi_d(T_4) = 4$.

4.2 Theorem: $\gamma(T_K) = \left\lfloor \frac{K+1}{2} \right\rfloor$ for $1 \leq k \leq 6$, k=8, k=10 and k=14

Proof: $\gamma(T_K) \geq \frac{|V(Tk)|}{1+\Delta(Tk)}$

$\gamma(T_K) \geq \left\lceil \frac{2k+1}{1+4} \right\rceil = \left\lceil \frac{2k+1}{5} \right\rceil$

Let S= $\{v_2, v_4, \ldots, v_{k+1}\}$ if k is odd and S= $\{v_2, v_4, \ldots, v_k\}$ if k is even.
In both cases, $|S| = \left\lfloor \frac{K+1}{2} \right\rfloor$ and S is a dominating set of T_K
Hence S is a minimum dominating set of $T_K = |S| = \left\lfloor \frac{K+1}{2} \right\rfloor$
$\left\lceil \frac{2k+1}{5} \right\rceil = \left\lfloor \frac{K+1}{2} \right\rfloor$ for $1 \leq k \leq 6$, k=8, k=10 and k=14

4.3 Theorem: $\chi_d(T_k) \leq \left\lfloor \frac{K+1}{2} \right\rfloor$

Proof: Contruct a $\left\lfloor \frac{K+1}{2} \right\rfloor$ colouring of T_k as follows
Let $V_i = \{v_{2i}\}$ for i=1,2,..., $\left\lfloor \frac{K+1}{2} \right\rfloor$
Assign colour $\left\lfloor \frac{K+1}{2} \right\rfloor + 1$to $u_1, u_2, u_3, \ldots, u_k$ and colour $\left\lfloor \frac{K+1}{2} \right\rfloor$+2 to v1,v3,v5,...
(i.e) $V_{\left\lfloor \frac{K+1}{2} \right\rfloor + 1} = \{u_1, u_2, u_3, u_k\}$ and

$$V_{\left\lfloor\frac{K+1}{2}\right\rfloor+2} = \begin{cases} \{v_1, v_3, v_{5,\ldots,} v_k\}, & if\ k\ is\ odd \\ \{v_1, v_2, \ldots, v_{k+1}\}, & if\ k\ is\ even \end{cases}$$

Clearly, it is a dominator colouring of T_k.

Hence, $\chi_d\ (T_k) \leq \left\lfloor\frac{K+1}{2}\right\rfloor$

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