SOME COMMON FIXED POINT THEOREM FOR TWO SELF MAPS UNDER CONE METRIC SPACE

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Abstract: The object of this paper is to obtain a common fixed point theorem for self mapping satisfying new contractive conditions in cone metric space , which generalizes and extend the result of solanki manoj etc. [16].

Keywords: Common Fixed Point, Self Mapping, Cone Metric Space, Rational Expression, Contractive Condition.

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1. Introduction: Fixed point theorems plays a basic role in applications of various branches of mathematics from calculus and linear algebra to topology and analysis. Much work has been done involving fixed point using the Banach Contraction Principle. This principle has been extended to other kind of contraction principle such as contraction conditions involving product rational expression and many others. The Banach contraction principle with rational expression have been extended and some fixed point and common fixed point theorems obtained in Banach[3] .

Banach contraction principle has been generalized by many mathematician Viz. , Abbas & Rhoades[1] , Bajaj[2] , Ciric[5] , Dubey & Pathak[6] , Fisher[7,8,9] , G. Emmanuell[10] , Imdad & Khan[11] , Jaggi[12] , Kannan[13] , Reich[14] , Rani D. and Chugh[15] , Solanki[16] , Singh S. L. Hematulin, Pant R[17] , Yadava R. N., Rajput S. S. & Bhardwaj[19] and many others. In the present paper we will find some fixed point & common fixed point theorems in complete metric space in rational expression.

2.0 Preliminary: Let G be a real Banach space and $'K'$ a subset at G. K is called a cone iff
a) K is closed, nonempty, and $K \neq \{0\}$
b) $a, b \in R, a, b \geq 0, \; x, y \in K \Longrightarrow ax + by \in K$.
c) $x \in K \; and \; -x \in K \Longrightarrow x = 0 \; i.e. \; K \cap (-K) = \{0\}$.
Given a cone $K \subset G$, we define a partial ordering $\leq$ with respect to
$K \; by \; x \leq y \; iff \; y - x \in K$
We write $x < y \; if \; x \leq y \; but \; x \neq y$.
While $x \ll y \; if \; y - x \in int \, K$.
The cone K is called normal if there is a number $M > 0 \; s.t. \, x, y \in G$.
$0 \leq x < y \; implies \; \| x \| \leq M \; \| y \|$.
The least positive number satisfying above is called the normal constant of.

Definition: 2.1 [2]: Let X be non-empty set G is a real Banach space and $K \subset G$, a cone. Suppose the mapping $d: X \times X \rightarrow G$ satifies
$d\,1.$ $0 < d\,(x,y) for \; all \; x, y \in X \; and \; d\,(x,y) = 0 \; iff \; x = y \quad and \; d\,(x,y) = 0 \; iff \; x = y$
$d\,2.$ $d\,(x,y) = d\,(y,x) for \; all \; x, y \in X$
$d\,3.$ $d\,(x,y) \leq d(x,z) + d(z,y) for \; all \; x, y, z \in X$
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 [4] Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.

3.0 Main Result:

Theorem 3.1: Let (X,d) *be a complete cone metric space and* K *be a normal cone with normal constant* M . *Let* $T : X \to X$ *be a*

$$d(Tx,Ty) \leq \alpha_1 \frac{d(x,Ty)d(y,Ty)+[d(x,y)]^2}{d(x,y)} +\alpha_2 \frac{d(x,Tx)d(x,Ty)+[d(x,y)]^2}{d(x,y)} +\alpha_3 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \alpha_4[d(x,Tx)+d(y,Ty)] +\alpha_5[d(y,Tx)+d(x,Ty)] + \alpha_6[d(x,y)] + L\min\{ d(x,Ty), d(y,Tx)\}. \quad \text{(3.1.a)}$$

for all $x, y \in X$, *where* $L \geq 0$ *and* $x \neq y$ *and for some* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in [0,1)$ *with* $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 < 1$. *Then* T *has, unique fixed point in* X.

Proof: Let X_0 be an arbitrary point in X. and we define a sequence $\{x_n\}$ be. means of iterates of T by satisfying $T^n x_0 = x_n$, *where n is positive integer.*

If for some n then the result is immediate. So let $x_n \neq x_{n+1}$ *for all n.*

Choose $x_0 \in X$, *Set* $x_1 = Tx_0$, $x_n = Tx_{n-1}$

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1})$$
$$\leq \alpha_1 \frac{d(x_n,Tx_{n-1})d(x_{n-1},Tx_{n-1}) + [d(x_n,x_{n-1})]^2}{d(x_n,x_{n-1})}$$
$$+\alpha_2 \frac{d(x_n,Tx_n)d(x_n,Tx_{n-1}) + [d(x_n,x_{n-1})]^2}{d(x_n,x_{n-1})}$$
$$+\alpha_3 \frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})}{d(x_n,x_{n-1})}$$
$$+\alpha_4[d(x_n,Tx_n) + d(x_{n-1},Tx_{n-1})$$
$$+\alpha_5[d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})] +\alpha_6[d(x_n,x_{n-1})] + L\min\{ d(x_n,Tx_{n-1}), d(x_{n-1},Tx_n)\}$$

$$= \alpha_1 \frac{d(x_n,x_n)d(x_{n-1},x_n) + [d(x_n,x_{n-1})]^2}{d(x_n,x_{n-1})}$$
$$+\alpha_2 \frac{d(x_n,x_{n+1})d(x_n,x_n) + [d(x_n,x_{n-1})]^2}{d(x_n,x_{n-1})}$$
$$+\alpha_3 \frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{d(x_n,x_{n-1})}$$
$$+\alpha_4[d(x_n,x_{n+1}) + d(x_{n-1},x_n)$$
$$+\alpha_5[d(x_{n-1},x_{n+1}) + d(x_n,x_n)] +\alpha_6[d(x_n,x_{n-1})] + L\min\{ d(x_n,x_n), d(x_{n-1},x_{n+1})\}$$
$$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)d(x_n,x_{n-1}) + (\alpha_4 + \alpha_5)\, d(x_n,x_{n+1})$$
$$d(x_{n+1},x_n) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}{1-\alpha_4-\alpha_5} d(x_n,x_{n-1})$$
$$Where\ k = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}{1-\alpha_4-\alpha_5} < 1$$

Since $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 < 1$

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… … … … … … … … … … … … ….

$$d(x_{n+1},x_n) \leq k^n d(x_1,x_0)$$

By the triangle inequality we have $m \geq n$.

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots\dots\dots\dots\dots + d(x_{m-1},x_m)$$
$$\leq (k^n + k^{n+1} + \dots\dots. + k^{m-1})d(x_1,x_0)$$
$$d(x_n,x_m) \leq \frac{k^n}{1-k} d(x_1,x_0)$$

Now $\left\|d(x_n,x_m)\right\| \leq M \frac{k^n}{1-k} \left\|d(x_1,x_0)\right\|$

$\to 0$ *if* $n,m \to \infty$

So the sequence $\{x_n\}$ *is a cauchy sequence in* X,
so by completeness of X *this sequence must be convergent in* X. *Now* w *is another point of* X

$$d(w,Tw) \le d(w,x_{n+1}) + d(x_{n+1},Tw)$$
$$d(w,Tw) \le d(w,x_{n+1}) + d(Tx_n,Tw)$$
$$\le d(w,x_{n+1}) + \alpha_1 \frac{d(x_n,Tw)d(w,Tw) + [d(x_n,w)]^2}{d(x_n,w)}$$
$$+\alpha_2 \frac{d(x_n,Tx_n)d(x_n,Tw) + [d(x_n,w)]^2}{d(x_n,w)}$$
$$+\alpha_3 \frac{d(x_n,Tx_n)d(w,Tw)}{d(x_n,w)} + \alpha_4[d(x_n,Tx_n) + d(w,Tw)]$$
$$+\alpha_5[d(w,Tx_n) + d(x_n,Tw)] + \alpha_6[d(x_n,w)] + L\min\{d(x_n,Tw),d(w,Tx_n)\}$$
$$d(w,Tw) \le d(w,x_{n+1}) + \alpha_1 d(x_n,w) + \alpha_2[d(x_n,x_{n+1}) + d(x_n,w)] + \alpha_4[d(x_n,x_{n+1})]$$
$$+\alpha_5[d(w,x_{n+1}) + d(x_n,w)] + \alpha_6[d(x_n,w)] + L\min\{d(x_n,w),d(w,x_{n+1})\}$$

so using the condition of normality of cone

$$\|d(w,Tw)\| \le$$
$$M\left(\|d(w,x_{n+1})\| + \alpha_1\|d(x_n,w)\| + \alpha_2[\|d(x_n,x_{n+1})\| + \|d(x_n,w)\|] + \alpha_4[\|d(x_n,x_{n+1})\|]\right.$$
$$\left.+\alpha_5[\|d(w,x_{n+1})\| + \|d(x_n,w)\|] + \alpha_6[\|d(x_n,w)\|] + L\min\{\|d(x_n,w),d(w,x_{n+1})\|\}\right)$$

As $n \to 0$, *we have*

$$\|d(w,Tw)\| \le 0. \text{ Hence } w = Tw, \quad w \text{ is a fixed point of } T.$$

Theorem 3.2: Let (X,d) *be a complete cone metric space and* K *be a normal cone with normal constant* M. *Let* $S,T : X \to X$ *be a*

$$d(Sx,Ty)$$
$$\le \alpha_1 \frac{d(x,Ty)d(y,Ty)+[d(x,y)]^2}{d(x,y)}$$
$$+\alpha_2 \frac{d(x,Sx)d(x,Ty) + [d(x,y)]^2}{d(x,y)}$$
$$+\alpha_3 \frac{d(x,Sx)d(y,Ty)}{d(x,y)} + \alpha_4[d(x,Sx) + d(y,Ty)]$$
$$+\alpha_5[d(y,Sx) + d(x,Ty)] + \alpha_6[d(x,y)] + L\min\{d(x,Sx),d(x,Ty),d(y,Sx)\}. \quad \text{............. (3.1.a)}$$

for all $x,y \in X$, *where* $L \ge 0$ *and* $x \ne y$ *and for some*
$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6 \in [0,1)$ *with*
$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 < 1$. *Then* S,T *be a unique fixed point in* X.

Proof: Let X_0 be an arbitrary point in X.
Choose $x_0 \in X$, *Set* $x_2 = Sx_1$, $x_1 = Tx_0$
s.t. $x_{2n+2} = Sx_{2n+1}$ *and* $x_{2n+1} = Tx_{2n}$

$$d(x_{2n+2},x_{2n+1}) = d(Sx_{2n+1},Tx_{2n})$$
$$\le \alpha_1 \frac{d(x_{2n+1},Tx_{2n})d(x_{2n},Tx_{2n}) + [d(x_{2n+1},x_{2n})]^2}{d(x_{2n+1},x_{2n})}$$
$$+\alpha_2 \frac{d(x_{2n+1},Sx_{2n+1})d(x_{2n+1},Tx_{2n}) + [d(x_{2n+1},x_{2n})]^2}{d(x_{2n+1},x_{2n})}$$
$$+\alpha_3 \frac{d(x_{2n+1},Sx_{2n+1})d(x_{2n},Tx_{2n})}{d(x_{2n+1},x_{2n})} + \alpha_4[d(x_{2n+1},Sx_{2n+1}) + d(x_{2n},Tx_{2n}) + \alpha_5[d(x_{2n},Sx_{2n+1}) + d(x_{2n+1},Tx_{2n})] +$$
$$\alpha_6[d(x_{2n+1},x_{2n})] + L\min\{d(x_{2n+1},Sx_{2n+1}),d(x_{2n+1},Tx_{2n}),d(x_{2n},Sx_{2n+1})\}$$

$$= \alpha_1 \frac{d(x_{2n+1},x_{2n+1})d(x_{2n},x_{2n+1}) + [d(x_{2n+1},x_{2n})]^2}{d(x_{2n+1},x_{2n})}$$
$$+\alpha_2 \frac{d(x_{2n+1},x_{2n+2})d(x_{2n+1},x_{2n+1}) + [d(x_{2n+1},x_{2n})]^2}{d(x_{2n+1},x_{2n})}$$
$$+\alpha_3 \frac{d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+1})}{d(x_{2n+1},x_{2n})}$$
$$+\alpha_4[d(x_{2n+1},x_{2n+2}) + d(x_{2n},x_{2n+1})$$
$$+\alpha_5[d(x_{2n},x_{2n+2}) + d(x_{2n+1},x_{2n+1})] + \alpha_6[d(x_{2n+1},x_{2n})]+$$
$$+ L\min\{d(x_{2n+1},x_{2n+2}),d(x_{2n+1},x_{2n+1}),d(x_{2n},x_{2n+2})\}$$

$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) d(x_{2n+1}, x_{2n}) + (\alpha_4 + \alpha_5)\, d(x_{2n+1}, x_{2n+2})$

$d(x_{2n+2}, x_{2n+1}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}{1 - \alpha_4 - \alpha_5} d(x_{2n+1}, x_{2n})$

$Where\ k = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}{1 - \alpha_4 - \alpha_5} < 1$

$Since\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 < 1$

… … … … … … … … … … … … ….

… … … … … … … … … … … … ….

By the triangle inequality we have $m \geq n$.

$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots \dots \dots \dots \dots \dots + d(x_{m-1}, x_m)$

$\leq (k^n + k^{n+1} + \dots \dots \dots . + k^{n+m-1}) d(x_1, x_0)$

$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_1, x_0)$

Now $\| d(x_n, x_m) \| \leq M \frac{k^n}{1-k} \| d(x_1, x_0) \|$

$\rightarrow 0\ if\ n, m \rightarrow \infty$

So the sequence $\{x_n\}$ *is a cauchy sequence in* X,

so by completeness of X *this sequence must be convergent in* X. *Now* w *is another point of* X

$d(w, Tw) \leq d(w, x_{2n+2}) + d(x_{2n+2}, Tw)$

$d(w, Tw) \leq d(w, x_{2n+2}) + d(Tx_{2n+1}, Tw)$

$\leq d(w, x_{2n+2}) + \alpha_1 \frac{d(x_{2n+1}, Tw) d(w, Tw) + [d(x_{2n+1}, w)]^2}{d(x_{2n+1}, w)}$

$+\alpha_2 \frac{d(x_{2n+1}, Tx_{2n+1}) d(x_{2n+1}, Tw) + [d(x_{2n+1}, w)]^2}{d(x_{2n+1}, w)}$

$+\alpha_3 \frac{d(x_{2n+1}, Tx_{2n+1}) d(w, Tw)}{d(x_{2n+1}, w)} + \alpha_4 [d(x_{2n+1}, Tx_{2n+1}) + d(w, Tw)]$

$+\alpha_5 [d(w, Tx_{2n+1}) + d(x_{2n+1}, Tw)] + \alpha_6 [d(x_{2n+1}, w)] +$

$L \min\{d(x_{2n+1}, Tx_{2n+1}), d(x_{2n+1}, Tw), d(w, Tx_{2n+1})\}$

$d(w, Tw) \leq d(w, x_{2n+2}) + \alpha_1 d(x_{2n+1}, w) + \alpha_2 [d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, w)] + \alpha_4 [d(x_{2n+1}, x_{2n+2})] +$

$\alpha_5 [d(w, x_{2n+2}) + d(x_{2n+1}, w)] + \alpha_6 [d(x_{2n+1}, w)] + L \min\{d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, w), d(w, x_{2n+2})\}$

so using the condition of normality of cone

$\| d(w, Tw) \| \leq M (\| d(w, x_{2n+2}) \| + \alpha_1 \| d(x_{2n+1}, w) \| + \alpha_2 [\| d(x_{2n+1}, x_{2n+2}) \| + \| d(x_{2n+1}, w) \|] +$
$\alpha_4 [\| d(x_{2n+1}, x_{2n+2}) \|]$

$+\alpha_5 [\| d(w, x_{2n+2}) \| + \| d(x_{2n+1}, w) \|] + \alpha_6 [\| d(x_{2n+1}, w) \|] +$

$L \min\{ \| d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, w), d(w, x_{2n+2}) \| \})$

As $n \rightarrow 0$, *we have*

$\| d(w, Tw) \| \leq 0$. *Hence* $w = Tw$, & *similarly* $w = Sw$.

therefore w *is a fixed point of* S & T.

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