VECTOR SPACE AND LINEAR MAPPING FOR ELECTRORHEOLOGICAL FLUID

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Abstract: Electrorheological fluids are smart materials whose rheological properties are controllable through the applications of an external electric or magnetic field. This rheological properties of ER fluid can be exploited in ERF devices for advanced technological applications. The optimal design of ERF devices requires proper mathematical modeling and basic governing equations. This paper presents the governing equations and mathematical theory for ER fluids. The main goal of this work is to formulate vector spaces and linear mapping for electrorheological fluid. Also in this paper matrix representation and inner product space for ER fluid is described.

Keywords: Electrorheological Fluid, Electric Field, Vector Space, Linear Mapping.

Introduction: Electrorheological (ER) fluids are special viscous fluids, consisting of solid particles dispersed in an insulating carrier fluid, and that are undergo significant changes in their mechanical and rheological properties when an electric or magnetic field is applied. This property of the ER fluid can be exploited in different kinds of technological applications. They have broad applications potential in dampers, actuators, clutches, valves, etc. Typically, the rheological properties of an ER , including viscosity , shear stress and internal arrangement, either increase or decrease with the applied magnetic field strength, depending on the size and properties of the particles dispersed in carrier fluids. Such kind of response takes place in millisecond scale.

This paper gives an overview of governing equations and mathematical theory, from material properties, Physics of ER fluid and potential real life applications. The physical models, which were proposed for explaining observed ER behavior, are introduced chronologically. Electrorheological fluids can be modeled in many ways. The continuum mechanics and numerical simulations taking into account the dynamics and art of the particles in carrier fluids. The behavior and properties of ER fluids is strongly influenced by non-homogeneous electric fields and in the most of the models electric field is treated as variable and that has to be determined for getting complete solution of the model.

Mathematical behavior of Electrorheological fluid can be described in different manner based on continuum mechanics, direct numerical simulation, dynamics and interaction of particles and microstructure of base fluid. In [7] the functional relation for electrorheological fluid are developed through generalized Lebesgue and generalized Sobolev space. In this paper, we carry out the modeling for ER fluid, which builds on idea of linear algebra.

Governing Equations for Electrorheological Fluid: The behavior of the electrorheological fluid is depends upon, density of fluid, velocity of fluid, velocity gradient, electric field, magnetic induction and absolute temperature and denoted by the symbols ρ, v, D, E, B, θ respectively.
Then the set of governing equations for material motion in presence of electro-magnetic field are given by [1],

$\dot{\rho} + \rho\, div\, v = 0 \quad \dots(1)$

$\rho\dot{v} - div.T = \rho f + f_e \quad \dots.(2)$

$\rho\dot{e} - k\Delta\theta = T.D + \dot{P}.\varepsilon + (P.\varepsilon)divv - \mathcal{M}.\dot{B} + \mathfrak{J}.\mathcal{E} + \rho r \quad \dots(3)$

$\varepsilon(P\otimes\epsilon + \mathcal{M}\otimes B) = 0 \quad \dots.(4)$

$$(T + \emptyset I). D + k\frac{|\nabla\theta|^2}{\theta} + \mathfrak{J}.\mathcal{E} \geq 0 \quad \ldots.(5)$$

$$div(D_e) = q_e \quad \ldots.(6)$$

$$curlE + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \quad \ldots.(7)$$

$$div\ B = 0 \quad \ldots.(8)$$

$$curlH = \frac{1}{c}\frac{\partial D_e}{\partial t} + \frac{1}{c}(\mathfrak{J} + q_e v) \quad \ldots.(9)$$

$$\frac{\partial q_e}{\partial t} + div\ (\mathfrak{J} + q_e v) = 0 \quad \ldots.(10)$$

Where f_e is given by

$$f_e = q_e E + \frac{1}{c}J \times B + \frac{1}{c}\left(\frac{dP}{dt} + (div.v)P\right) \times B + \frac{1}{c}v \times ([\nabla B]P) + [\nabla B]^T\mathcal{M} + [\nabla E]P \quad \ldots(11)$$

And where the thermodynamic pressure $\emptyset$ is defined as

$$\emptyset = \rho^2\frac{\partial\psi}{\partial\rho} \quad \ldots.(12)$$

Where $\rho, v, D, E, B, \theta, \nabla\theta$ are assumed to be independent variables and $e, \psi, T, q, \mathcal{M}, P, \mathfrak{J}$ are the constitutive relations of the form
$f = \hat{f}(\rho, v, D, E, B, \theta, \nabla\theta)$

The equations (1), (2), (3).......... and (12) which describes the motion of the electrorheological material in electric or magnetic field. The equations that govern rheological behavior of ER fluid are linear in the electric, magnetic, and related fields. In these equations, the operations that are performed on the fields, differentiations and integrations, are linear operations. Because of this linearity, linear algebra is a very useful tool in electrorheological materials. It is useful in helping us develop good approximate representations of electrorheological fluid that are suitable for mathematical formulations.

Vector Spaces for Electrorheological Fluid:

Vector Space for ER Fluid: Let V be a non empty set of electrorheological fluid with two operations:
i) Vector addition: This assigns to any $u, v \in V$ a sum $u + v$ in V.
ii)Scalar Multiplication: This assigns to any $u \in V, k \in K$ a product $ku \in V$.
Then V is called a vector space for electrorheological fluid (over the field K) if the following axioms hold for any vectors $u, v, w \in V$:
[A1] $(u + v) + w = u + (v + w)$
[A2] There is a vector in V, denoted by 0 and called the zero vector, such that, for any $u \in V$,
$u + 0 = 0 + u = u$
[A3] For each $u \in V$, there is a vector in V, denoted by $-u$ and called the negative of u , such that ,
$u + (-u) = (-u) + u = 0$
[A4] $u + v = v + u$
[M1] $k(u + v) = ku + kv$ for any $k \in K$
[M2]$(a + b)u = au + bu$ for any $a, b \in K$
[M3]$(ab)u = a(bu)$ for any $a, b \in K$
[M4]$1u = u$ for the unit scalar $1 \in K$

The above axioms naturally split into two sets. The first four are only concerned with the additive structure of V. On the other hand, the remaining four axioms are concerned with the action of the field of scalars on the vector space V.

Scalar Field for ER Fluid: Suppose K is a nonempty set of ER fluid equipped with two binary operations called addition and multiplication and denoted by + and – respectively i.e. for all $a, b \in K$ we have $a + b \in K$ & $a.b \in K$.
Then the algebraic structure $(K, +, \ .)$ is called a scalar field for ER fluid, if the following postulates are satisfied,
[K1] $a + b = b + a \ for\ all\ a, b \in K$
[K2] $(a + b) + c = a + (b + c)\ for\ all\ a, b, c \in K$
[K3] there exits element a in K such that $a + 0 = a\ for\ all\ a \in K$
[K4] to each element a in K there exits an element –a in K such that $a + (-a) = 0$
[K5] $a.b = b.a\ \ for\ all\ a, b \in K$
[K6] $(a.b).c = a.(b.c)\ for\ all\ a, b, c \in K$
[K7] there exits element a non zero element denoted by 1 in K such that $a.1 = a\ \ for\ all\ a \in K$
[K8] to every non zero element a in K there exits an element a^{-1} in K such that $a + a^{-1} = 0$
[K9] $a.(b + c) = a.b + a.c\ for\ all\ a, b, c \in K$

Linear Combination, Spanning Sets: Let V be a vector space for ER fluid over the field K. A vector u in V is a linear combination of vectors $u_1, u_2, u_3, \dots., u_n$ in V if there exits scalars $a_1, a_2, a_3, , \dots, a_n$ such that
$v = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n$

Let V be a vector space for ER fluid over the field K. A vector $u_1, u_2, u_3, \dots., u_n$ in V are said to span V or to form a spanning set of V if every v in V is a linear combination of the vectors $u_1, u_2, u_3, \dots., u_n$, that is, if there exits scalars $a_1, a_2, a_3, , \dots, a_n$ in K such that
$v = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n$

The set of linear span is generally denoted by S
Hence $S = \{u_1, u_2, u_3, \dots., u_n\}$
The collection of all such linear combinations denoted by L(S)
Thus, we have
$L(s) = \{a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n: u_1, u_2, u_3, \dots., u_n \in V\ \&\ a_1, a_2, a_3, , \dots, a_n \in K\}$

Subspaces for ER Fluid: Let V be a vector space for ER fluid over the field K and let $W \subseteq V$. Then W is called a subspace of V if W itself is a vector space over K with respect to the operations of vector addition and scalar multiplication on V.

The way in which one shows that any set W is a vector space is to show W satisfies the eight axioms of a vector space. However if W is a subset of a vector space V, then some of the axioms automatically hold in W , since they already holds in V . Simple criteria for identifying subspaces follow.

If W is a subset of a vector space V. Then W is subspace of V if the following two conditions hold:
i) The zero vector 0 belongs to W.
ii) For every $, v \in W, k \in K$: a) the sum $u + v \in W$ b) the multiple $ku \in W$

Linear Dependence and Independence: The notions of linear dependence and linear combinations are closely related.
Let V be a vector space of ER fluid. A finite set $\{v_1, v_2, \dots., v_n\}$ of vector space V is said to be linearly dependent if there exits scalars $a_1, a_2, a_3, , \dots, a_n \in K$ not all them 0 such that
$a_1v_1 + a_2v_2 + a_2v_3 + \cdots + a_nv_n = 0$
Let V be a vector space of ER fluid. A finite set $\{v_1, v_2, \dots., v_n\}$ of vector space V is said to be linearly independent if every relation of the form
$a_1v_1 + a_2v_2 + a_2v_3 + \cdots + a_nv_n = 0\ , a_i \in K, 1 \le i \le n$
$\Rightarrow a_i = 0\ for\ each\ \ 1 \le i \le n$
Any infinite set of vectors of V is said to be independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Basis and Dimension of Electrorheological Vector Space: A subset S of a vector space V of ER fluid is said to be a basis of V if
i) S consists of linearly independent vectors.
ii) S generates V i.e., L(S)=V i.e. , each vector in V is a linear combination of a finite number of elements of S
A vector space V is said to be finite dimensional if there exists a finite subset S of V such that V=L(S). The dimension of vector space V is denoted by $dimV$

Linear Mappings or Linear Transformations for Electrorheological Fluid: Let U be the vector space for the electrorheological fluid having independent vectors $\rho, v, D, E, B, \theta, \nabla\theta$ and V be another vector space for the same electrorheological fluid having independent variable $e, \psi, T, q, \mathcal{M}, P, \mathfrak{I}$ over the same field K. A mapping $T: U \rightarrow V$ is called a linear mapping or linear transformation for ER fluid if is satisfies the following condition
i) $T(\alpha+\beta) = T(\alpha) + T(\beta)$ for all $\alpha, \beta \in U$ and for all $a \in K$.
ii) $T(a\alpha) = aT(\alpha)$ for all $\alpha \in U$ and for all $a \in K$.
The condition (i) and (ii) can be combined into single condition
$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$for all $\alpha, \beta \in U$ and for all $a, b \in K$.

Isomorphism of Electrorheological Vector Space: Let U and V be the vector space for the electrorheological fluid over the same field K. Then a mapping $T: U \rightarrow V$ is called a isomorphism of U onto V if
i) T is one one
ii) T is onto
iii)$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$for all $\alpha, \beta \in U$ and for all $a, b \in K$.
Then the vector space V is also called the isomorphic image of the vector space U.

Kernel and Image of a Linear Mapping: Let $T: U \rightarrow V$ is a linear mapping for ER fluid. The kernel of T, written KerT, is the set of elements in U that map into the zero vector o in V; i.e.
$KerT = \{u \in U: T(u) = 0\}$
The image or range of T, written ImT, is the set of image points in V; i.e.
$ImT = \{v \in V: \exists\, u \in U \text{ such that } T(u) = v\}$

Rank and Nullity of a Linear Mapping: Let $T: U \rightarrow V$ is a linear mapping for ER fluid. Then the rank of the T is defined to be the dimension of its range, and the nullity of T is defined to be the dimension of its kernel.
$Rank(T) = \dim(ImT)$ & $Nullity(T) = \dim(KerT)$

Matrix Representation of ER Fluid: Let U be an n dimensional vector space of electrorheological fluid over the field K and let V be an m dimensional vector space of electrorheological fluid over the field K. Let $\beta = \{u_1, u_2, \dots, u_n\}$ and $\beta' = \{v_1, v_2, \dots, v_m\}$ be ordered bases for U and V respectively. Let T is linear mapping from U to V.
We know that T is completely determined by its action on the vectors u_j belonging to a basis of for. Each of the n vectors $T(u_j)$ is uniquely expressed as a linear combination $\beta' = \{v_1, v_2, \dots, v_m\}$ because $T(u_j) \in V$ and these m vectors forms a basis for V.
Let for j=1, 2,..., n.

$$T(u_j) = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{mj}v_n = \sum_{i=1}^{m} a_{ij}v_i$$

The scalars a_{ij} are the coordinates of $T(u_j)$ in the ordered basis β' . The $m \times n$ matrix whose jth column (j=1, 2,..., n) consists of these coordinates is called the matrix of the linear transformation of T relative to the pair of ordered basis β *and* β'. We shall denote it by the symbol $[T; \beta; \beta']$ or simply by T. Thus

$$T = [T; \beta; \beta'] = Matrix\ of\ T\ relative\ to\ ordred\ basis\ \beta\ \&\ \beta' = [a_{ij}]_{m\times n}$$

Where

$$T(u_j) = \sum_{i=1}^{m} a_{ij} v_i \quad for\ each\ j = 1,2,\dots,n$$

The coordinates of $T(u_1)$ in ordered basis β' form the first column of this matrix, the coordinates $T(u_2)$ in ordered basis β' form the second column of this matrix and so on.
The $m \times n$ matrix $\left[a_{ij}\right]_{m\times n}$ completely determines the linear transformation T through the above formula. Therefore, the matrix $\left[a_{ij}\right]_{m\times n}$ represents the transformation matrix for electrorheological fuid.

Change of Basis for ER Fluid: Let $\beta = \{u_1, u_2, \dots, u_n\}$ be a basis of electrorheological vector space and $\beta' = \{v_1, v_2, \dots, v_n\}$ be another basis for same electrorheological vector space. Since β is a basis of electrorheological vector space, each vector in the β' can be written uniquely as a linear combination of the vectors in β.

$$\begin{gathered} v_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ v_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ v_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n \end{gathered}$$

Let A be the transpose of the above matrix of coefficients; that is,
Let $A = [A_{ij}]_{n\times n}$ Where $A_{ij} = a_{ji}$.
Then A is called the change of basis matrix or transition matrix from the old basis β to the new basis β' .

Inner Product Space for ER Fluid:
Definition: Let V is the vector space for electrorheological fluid over the field K. Here we first restrict K to be the real field R, in which case V is called a real vector space, in this paper, we extend our results to the case where K is the complex field C, in which case V is called complex vector space.
Let V be a vector space for electrorheological fluid over the field K, where K is either the field of real numbers or the field of complex numbers. An inner product on V is a function from $V \times V$ into K which assigns to each ordered pair of vectors α, β in V a scalar $\langle \alpha, \beta \rangle$ in such a way that
i) Symmetric Property: $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$
(Here $\overline{\langle \beta, \alpha \rangle}$ denotes the conjugate complex number of the number $\langle \alpha, \beta \rangle$)
ii) Linearity Property: $\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$
iii) Positive Definite Property: $\langle \alpha, \alpha \rangle \geq 0\ and\ \langle \alpha, \alpha \rangle = 0 \Rightarrow \alpha = 0$
for any $\alpha, \beta, \gamma \in V\ and\ a, b \in K$.
Also the vector space V is then said to be an inner product space with respect to the specified inner product defined on it.

Norm or Length of a Vector in an Inner Product Space: Consider the vector space V for electrorheological fluid over the field R with standard inner product space on it. If $\alpha = (a_1, a_2, a_3) \in V(R)$, we have
$\langle \alpha, \alpha \rangle = a_1^2 + a_2^2 + a_3^2$
Now we know that in the three dimensional Euclidean space $\sqrt{(a_1^2 + a_2^2 + a_3^2)}$ is the length of a vector $\alpha = (a_1, a_2, a_3) \in V(R)$. Taking motivation from this fact, we make the following definition.

Definition: Let V be an inner product space. If $\alpha \in V$, then norm or length of the vector α , written$\|\alpha\|$, is defined as the positive square root of $\langle \alpha, \alpha \rangle$ that is
$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$
In an inner product space a vector is called a unit vector if its length is 1.
Let V be an inner product space. The two vectors $\alpha, \beta \in V$ are said to be orthogonal and α is said to be orthogonal to β if
$\langle \alpha, \beta \rangle = 0$

Conclusion: In this paper, linear algebra has been developed for ER fluid to determine constitutive relations for dependent variables like internal specific energy, potential function, yield stress, charge on particles, stress tensor, magnetic & electric polarization, heat flux. It also focuses on influence of electric or magnetic field on art of electrorheological fluid. In future we shall derive constitutive relations for ER fluid and compare them with experimental results that are available.

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