GEODETIC DISTANCE 2-DOMINATION OF GRAPHS

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Abstract: A set S of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex $v \in V(G)$ is an element of S or adjacent to an element of S. The minimum cardinality of a dominating set is the domination number of G and is denoted by $\gamma(G)$. A set of vertices S of a graph G is a distance 2-dominating set of G if the distance between each vertex $u \in (V(G) - S)$ and S is at most two. Let $\gamma_2(G)$ denote the size of a smallest distance 2-dominating set of G. Let $I(u, v)$ be the set (interval) of all vertices lying on some u–v geodesic of G, and for a nonempty subset S of $V(G), I(S) = \cup_{u,v\in S} I(u, v)$. A set $S \subseteq V$ is a geodetic set of G if $I[S] = V(G)$. The minimum cardinality of a geodetic set of G is the geodetic number of G and is denoted by $g(G)$. A distance 2 – dominating set is called a geodetic distance 2-dominating set if S is also a geodetic set. The minimum cardinality of geodetic distance 2-dominating set is the geodetic distance 2-domination number and is denoted by $\gamma_{2\,g}(G)$. In this paper we study the geodetic distance 2-domination number of graphs.

Keywords: Geodetic set, Dominating set , Geodetic dominating set, Distance 2-dominating set.

Introduction: We consider a finite graph without loops and multiple edges. Let G be a graph the vertex set $v(G)$ and the edge set $E(G)$.The *order* of the graph G is $|V(G)|$ and *size* is $|E(G)|$. The *degree* $d(v)$ of the vertex $v\epsilon V(G)$ is the number of the edges adjacent to v i.e., $d(v) = |N(v)|$. For a vertex $v \epsilon V(G)$, the open neighborhood $N(v)$ is the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote the maximum and minimum degree of the graph G respectively.

If G is a graph then its complement is denoted by $\overline{G}$. The *girth* of a graph G is the length of the shortest cycle. The *Triangle free graph* is an undirected graph in which no three vertices form a triangle of edges.In a graph G a vertex x is simplicial if its neighborhood $N(x)$ induces a complete subgraph of G. If G is a connected graph, then the *distance* $d(x, y)$ is the length of a shortest x -y path in G. The *diameter* of a connected graph G is defined by $diam(G) = max_{x,y\in V(G)}\, d(x, y)$. A $x - y$ path of length $d(x, y)$ is called a $x - y$ *geodesic*. A vertex v is said to lie on an x-y geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying on some x-y geodesic of G while for $S \subseteq V(G)\; I[S] = \cup_{x,y\in S} I[x, y]$. A set $S \subseteq V$ is a geodetic set of G if $I[S] = V(G)$. The minimum cardinality of a geodetic set of G is the geodetic number of G and is denoted by $g(G)$. A set $S \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is a dominating set of G if every vertex $v \subseteq V(G)$ is an element of S or adjacent to the element of S. A set of vertices S of a graph G is a distance 2-dominating set of G if the distance between each vertex $u \in (V(G) - S)$ and S is at most two. Let $\gamma_2(G)$ denote the size of a smallest distance 2-dominating set of G. A set S is called geodetic dominating set of G if S is both geodetic set and a dominating set of G. The minimum cardinality of geodetic dominating set is the geodetic domination number and is denoted by $\gamma_g(G)$. A distance 2 – dominating set is called a geodetic distance 2-dominating set if S is also a geodetic set. The minimum cardinality of geodetic distance 2-dominating set is the geodetic distance 2-domination number. It is denoted by $\gamma_{2\,g}(G)$.

Geodetic Distance 2-Domination:

Observation 2.1: $\gamma_{2\,g}(G) = n$ *iff* G *is the complete graph on n vertices.*

Observation 2.2: *If G is a connected graph of* order $n \geq 2$ then,

$$2 \leq \max\{g(G), \gamma_2(G)\} \leq \gamma_{2\,g}(G) \leq n.$$

Observation 2.4: *If G is cycle of order n then* $\gamma_{2\,g}(C_n) = \left\lceil \frac{n}{5} \right\rceil$ for $n \geq 11$.

Theorem 2.3: *Let G be a connected graph of order ≥ 2 . Then $\gamma_{2\,g}(G) = 2$ iff there exist a geodetic set $S = \{u, v\}$ of G such that $(u, v) \leq 5$.*

Proof: Given $\gamma_{2\,g}(G) = 2$. If $d(u, v) > 5$,
then there exist a vertex x in the u-v path such that it is not distance 2-dominated either by u or v which is a contradiction to the fact that $\gamma_{2\,g}(G) = 2$. Hence $d(u, v) \leq 5$. Conversely if $d(u, v) \leq 5$ and $S = \{u, v\}$ is a geodetic set then every vertex in $u - v$ path has distance at most two form S.

Theorem 2.5: *If G is path of order n then the geodetic distance 2-domination number is given by $\gamma_{2\,g}(P_n) = \left\lceil \frac{n+4}{5} \right\rceil$ for all n.*

Lemma 2.6: *If G is a connected graph with $\gamma_2(G) = 1$ then $\gamma_{2\,g}(G) = g(G)$.*

Proof: For $G = K_n$, then $\gamma_2(G) = 1$ and $\gamma_{2\,g}(G) = n$. Now consider $G \neq K_n$ therefore $\Delta(G) = n - 1$. Since $\neq K_n$, G has at least two non-adjacent vertices and so $diam(G) \leq 5$. Let S be a minimum geodetic set of G. Let x is not a vertex of S. Since S is a geodetic set there exist two vertices $u, v \in S$ such that x belongs to a $u - v$ geodesic. Since $diam(G) \leq 5$, $d(u, x) \leq 2$
or $d(v, x) \leq 2$. Hence S is a geodetic 2-dominating set. Hence $\gamma_{2\,g}(G) \leq |S| = g(G)$. Also $\gamma_{2\,g}(G) \geq g(G)$. Therefore $\gamma_{2\,g}(G) = g(G)$.

Observation 2.7: *For complete bipartite graph $K_{m,n}$ with $\gamma_{2\,g}(K_{m,n}) = min\{m, n\}$.*

Preposition 2.8 [1]: If G is a connected graph of order $n \geq 2$, then $\gamma_{2\,g}(G) \leq n - \left\lfloor \frac{4diam(G)}{5} \right\rfloor$.

Proof: Let $diam(G) = d = 5t + r$ with integers r, t such that $0 \leq r \leq 4$, and let u_0 and u_d be two vertices in G such that $d(u_0, u_d) = d$. Let P={ $u_0 u_1 \dots u_d\}$ be a $u_0 - u_d$ shortest path. Let $A = \{u_0, u_5, \dots u_{5t}, u_{5t+r}\}$. Clearly $D = V(G)\backslash(V(P)\backslash A)$ is a geodetic distance 2-dominating set of G. $|A| = t + 1$ when $r = 0$ and $|A| = t + 2$ when $1 \leq r \leq 4$, then we can find

$$
\begin{aligned}
|V(P)\backslash A| &= \left\lfloor \frac{4(5t + r)}{5} \right\rfloor \\
&= \left\lfloor \frac{4diam(G)}{5} \right\rfloor
\end{aligned}
$$

And hence the Theorem.

Complexity of the Geodetic Distance 2-Domination Problem:

Theorem 3.1: *The Geodetic Distance 2-Dominating Problem is NP-Complete.*

Proof: The dominating set problem is a well known NP-complete problem. The proof's reduction of Geodetic distance 2-dominating problem is from the same. Let $G = (V, E)$ be a graph. Now we are going to construct the graph G' as follows. $G' = G \cup K_n \cup G''$ where G is the given graph, K_n is a complete graph of order n and G'' is an independent set of pendent vertices.

The Graph G' consist of three layers: the top layer is the graph, the middle layer is the complete graph and the bottom layer is the independent set of pendent vertices.

If X is a geodetic distance 2-dominating set of G' , then there exists a geodetic distance 2-dominating set Y with $|Y| \leq |X|$, such that $Y = S \cup V''$ and $S \subseteq V$ where V and V'' are the vertex set of G and G'' respectively.

Let S be a dominating set of G and for the vertex set $S \cup G''$ in G' the set of paths is defined by

$$Y' = \{xyy'y'': x \in S, xy \in E\}$$

If we consider the independent set of vertices in G'' and the dominating set of G, then all the vertices are distance 2-dominated. Hence it is clear that G' is geodetic distance two dominated.

Conversely, suppose that $S \cup G''$ is a geodetic distance 2-dominating set of G'. Then $I'(S \cup G'')$ covers all the vertices of G' in geodesic. Suppose a vertex $y \in V\backslash S$. Since $(S \cup G'')$ is a geodetic distance 2-dominating set y must be adjacent to at least one vertex from S or else y would not be covered by $I'(S \cup G'')$. This implies that S is a dominating set of G.

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