ON THE DIOPHANTINE EQUATION $(p^k-1)^x+(p^k)^y=z^2$

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Abstract: In this paper, we apply Catalan's conjecture to find the solutions of the Titled equation when $p=2$ and $p=3$, where x, y, z are non-negative integers and k is positive integer.

Keywords: Diophantine Equations, Exponential Diophantine Equation.

Introduction: The Diophantine equation of the form

$$p^x+(p+1)^y=z^2 \qquad (1)$$

is studied by many researchers. In 2013, B. Sroysang[1] studied the Diophantine equation $7^x+8^y=z^2$ and posed an open problem to find all the solutions of the Equation (1). In 2013, Somchit Chotchaisthit [3] proved that Equation (1) with p a Mersenne Prime has only two solutions $(p,x,y,z)=(7,0,1,3)$ and $(p,x,y,z)=(3,2,2,5)$, however it is easy to check that $(p,x,y,z)=(3,1,0,2)$ is also it's solution.

In 2015, A. Suvernamani, Tawana Ampawa [4] studied Titled equation for $p=2$ and k an odd integer at least 3. In the same year, Azizul Hoque and Himashree Kalita [5], studied the titled equation for $p=2$ and $k>1$ and claimed to have only three solutions. But we see that there are more solutions to this equation.

In this paper, we give all the solutions of the Titled equation for $p=2$ and $p=3$ where x,y,z are non-negative integers and k is positive integer.

Preliminaries: We use the following proposition.

Proposition 1: (Catalan's Conjecture): (3,2,2,3) is a unique solution (a,b,x,y) of the Diophantine equation $a^x-b^y=1$ where a,b,x and y are integers with $\min\{a,b,x,y\}>1$.

Proof: See in [2].

Main Results: We prove the results in two theorems.

Theorem 1: Consider the equation

$$(2^k-1)^x+(2^k)^y=z^2 \qquad (2)$$

where x,y,z are non-negative integers and k is positive integer. All the solutions of Equation (2) are given by $(x,y,z,k)\,\epsilon\, A\cup B\cup C$ where $A=\{(m,3,3,1)/m\epsilon\mathbb{N}\}$, $B=\{\,(1,0,2^n,2n)\,/\,n\,\epsilon\mathbb{N}\,\}$ and $C=\{(0,1,3,3),(0,3,3,1),(2,2,5,2)\}$

Proof: We prove the theorem by taking following cases. Clearly $z>1$

Case : $x=0$

Equation (2) becomes $1+2^{ky}=z^2$, i.e. $z^2-2^{ky}=1$.

If $ky\,\epsilon\,\{0,1\}$, then there is no solution.

If $ky > 1$, then by Catalan's conjecture, $z = 3$ and $ky = 3$. Hence the solutions are $(x, y, z, k) = (0,1,3,3)$ and $(x, y, z, k) = (0,3,3,1)$.

Case 2: $x = 1$
Equation (2) becomes $(2^k - 1) + 2^{ky} = z^2$
Subcase 2.1 : $y = 0$. Then we get $2^k = z^2$ which is possible only if k is even say $= 2n$. Then $z = 2^n$. Hence the solutions are $(x, y, z, k) = (1,0,2^n, 2n)$, $n \epsilon \mathbb{N}$.
Subcase 2.2 : $y = 1$. Then we get $2^{k+1} - z^2 = 1$. By Proposition (1), there is no solution.
Subcase 2.3 : $y > 1$. We get

$$2^k - 1 + 2^{ky} = z^2. \tag{3}$$

If $k = 1$, then $z^2 - 2^y = 1$ and by Proposition (1), we get the solution $(x, y, z, k) = (1,3,3,1)$.
Let $k > 1$, taking Equation (3) modulo 4, we get $z^2 \equiv -1(mod\ 4)$ which is not possible.

Case 3: $x > 1$
Subcase 3.1 : $y = 0$. We get the equation $z^2 - (2^k - 1)^x = 1$. This has no solution for $k = 1$. If $k > 1$, then by Catalan's conjecture $2^k = 3$ which is not possible.
Subcase 3.2 : $y \geq 1$.
If $k = 1$, then $z^2 - 2^y = 1$, by Catalan's conjecture we have the solutions $(x, y, z, k) = (m, 3,3,1)$ where m is positive integer greater than 1. If $y = 0$ or 1, there is no solution.
Let $k > 1$. We have $z^2 \equiv 1(mod\ 4)$ so that $(2^k - 1)^x \equiv 1(mod\ 4)$ which is possible only if x is even, say $x = 2l$. Then $2^{ky} = z^2 - (2^k - 1)^{2l} = (z + (2^k - 1)^l)(z - (2^k - 1)^l)$. There are α and β such that $2^\alpha = z + (2^k - 1)^l$ and $2^\beta = z - (2^k - 1)$, where $\alpha + \beta = ky$, $\alpha > \beta$. Now $2^\beta(2^{\alpha-\beta} - 1) = 2(2^k - 1)^l$ which implies $\beta = 1$ and $2^{\alpha-\beta} - 1 = (2^k - 1)^l$ and thus

$$2^{\alpha-1} - (2^k - 1)^l = 1 \tag{4}$$

If $\min\{\alpha - 1, l\} > 1$, then by Catalan's conjecture there is no solution.
If $\alpha = 1$, then $ky = 2$ which gives $y = 1$ and $k = 2$ and thus from Equation (4) we get $1 - (3^l) = 1$ which is not possible.
If $\alpha = 2$, then $ky = 3$ which gives $y = 1$ and $k = 3$ and thus from Equation (4) we get $2 - (7^l) = 1$ which is not possible.
If $l = 1$, from Equation (4), $2^{\alpha-1} = 2^k$ which gives $\alpha - 1 = k \Rightarrow ky - 1 = k + 1$ and thus we get $k(y - 1) = 2$ which is possible only if $k = 2$ and $y = 2$. Here we get the solution $(x, y, z, k) = (2,2,5,2)$.
Hence the proof.

Now we find the solutions of Titled equation when $p = 3$ in next theorem

Theorem : Consider the equation

$$(3^k - 1)^x + (3^k)^y = z^2 \tag{5}$$

where x, y, z are non-negative integers, x is not the multiple of 4 and k is positive integer. All the solutions of Equation (5) are given by $(x, y, z, k) \epsilon \{(1,0,3^n, 2n) / n \epsilon \mathbb{N}\} \cup \{(0,1,2,1), (3,0,3,1)\}$.
Proof: Clearly $z > 1$

Case : $x = 0$
Equation (5) becomes $1 + 3^{ky} = z^2$, i.e. $z^2 - 3^{ky} = 1$.
If $ky > 1$, then by Catalan's conjecture, there is no solution. Clearly for $y = 0$ there is no solution.
Let $k = 1$ and $y = 1$ then we get $z^2 = 4$ which gives $z = 2$ and the solution is $(x, y, z, k) = (0,1,2,1)$

Case 2: $x = 1$
Equation (5) becomes $(3^k - 1) + 3^{ky} = z^2$
Subcase 2.1 : $y = 0$. Then we get $3^k = z^2$ which is possible only if k is even say $= 2n$. Then $z = 3^n$. Hence the solutions are $(x, y, z, k) = (1,0,3^n, 2n)$, $n\epsilon\mathbb{N}$.
Subcase 2.2 : $y \geq 1$. We have $(3^k - 1) + 3^{ky} = z^2$. Taking this equation modulo 3, we get $z^2 \equiv 2(mod\ 3)$ which is not possible.

Case 3: $x > 1$
Subcase 3.1 : $y = 0$. We get the equation $z^2 - (3^k - 1)^x = 1$. By Proposition (1), we get $z = 3$, $x = 3$ and $k = 1$. Thus, the solution is $(x, y, z, k) = (3,0,3,1)$.

Subcase 3.2 : $y \geq 1$.Taking Equation (5) modulo 3, we get $(-1)^x \equiv z^2(mod\ 3)$ which is possible only if x is even say $x = 2m$. Now

$$3^{ky} = z^2 - (3^k - 1)^{2m} = (z + (3^k - 1)^m)(z - (3^k - 1)^m)$$

There are α and β such that $3^\alpha = z + (3^k - 1)^m$ and $3^\beta = z - (3^k - 1)$, where $\alpha + \beta = ky$, $\alpha > \beta$. Now $3^\beta(3^{\alpha-\beta} - 1) = 2(3^k - 1)^m$. Now if $\beta \neq 0$,then $0 \equiv 2(-1)^m(mod\ 3)$ which is not possible.
Let $\beta = 0$, then $3^\alpha - 1 = 2(3^k - 1)^m$. Taking this equation modulo 3, we get

$$0 \equiv (2(-1)^m + 1)(mod\ 3)$$

This is possible only if m is even. But then x will be a multiple of 4. Hence there is no solution in this case.
Hence the proof.

Conclusion: In this paper, we found all the solutions of the Diophantine equation $(2^k - 1)^x + (2^k)^y = z^2$. Also we found all the solutions of $(3^k - 1)^x + (3^k)^y = z^2$ except the case when x is a multiple of 4. Thus, to find the solutions if $x = 4$ is an open problem and also to find all the solutions of $(p^k - 1)^x + (p^k)^y = z^2$ when p is prime other than $p = 2$ and $p = 3$ is still an open problem.

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