A STUDY ON HARMONIOUS COLORING OF CERTAIN CLASSES OF GRAPHS WITH DIAMETER ATMOST 3

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Abstract: Given a simple graph G, a harmonious coloring of a simple graph is the proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number of G, denoted by χ_h is the least number of colors in a harmonious coloring of G. In this paper we have determined the harmonious chromatic number of cyclic split graph, crown graph and central graph of cyclic split graph.

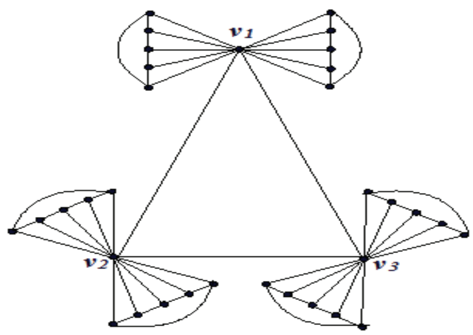
Keywords: Crown Graph, Cyclic Split Graph, Central Graph, Harmonious Coloring.

Introduction: Graph theory plays an important role in several areas of computer science, such as switching theory and logic design, computer graphics, operating systems, compiler and information organization and retrieval. In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation etc. Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. Graph coloring is a special case of graph labeling; it is an assignment of labels traditionally called colors to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph. A proper vertex coloring or simply a vertex coloring of a graph is an assignment of colors to the vertices of G such that no two adjacent vertices share the same color. The minimum number of colors used to color the graph G is called the chromatic number and is denoted by $\chi(G)$. A harmonious coloring of a simple graph G is a proper coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_h(G)$ is the least number of colors in such a coloring. This parameter was introduced by Miller and Pritikin. Every graph has a harmonious coloring, since it suffices to assign every vertex a distinct color; thus $\chi_h(G) \leq |V(G)|$[2]. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard[1]. Edwards investigated the harmonious chromatic number problem for almost all trees, bounded degree trees[3]. There are only a few families for which we can have exact solutions in polynomial time (including paths, cycles, unions of paths and cycles, stars, complete graphs, complete bipartite graphs and threshold graphs). Motivated by these we have determined the harmonious chromatic number of cyclic split graph, crown graph and central graph of cyclic split graph. We have also determined the maximum matching number of these graphs.

Preliminaries: A graph G is an ordered triple $V(G), E(G), \psi(G)$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$ of edges, and a mapping $\psi(G) : E(G) \rightarrow V(G) \times V(G)$ called an incidence function which maps an edge into a pair of vertices called end-vertices of the edge. A complete graph of order n is a simple graph with n vertices in which every vertex is adjacent to every other. The complete graph on n vertices is denoted by K_n. A clique of a graph G is a set $S \subseteq V$ that induces a complete subgraph of G. A set $S \subseteq V$ is said to be independent if no two vertices of S are adjacent in G. A split graph is a graph G in which the vertex set can be split into two sets K and S, such that K induces a clique and S an independent set in G. The central graph of any graph G is obtained by subdividing each edge of G exactly once and joining all the non adjacent vertices of G. A matching in a simple graph G is a set of edges with no shared endpoints. The vertices incident to the edges of a matching M are said to be saturated by M; the others are unsaturated. A perfect matching in a graph is a matching that saturates every vertex. A perfect matching in a graph is a matching that saturates every vertex. A maximum matching is a matching of maximum size among all matching's in the graph, denoted by $\alpha'(G)$.

Main Results:

Definition 3.1: A cyclic split graph is a split graph in which the vertices can be partitioned into a clique and an independent set of cycles. Thus, we consider a cyclic split graph $C_rK_n^k$ which has a complete graph K_n with vertices $v_1, v_2, \dots \dots, v_n$ and kn wheels $W_{i,j}$ attached at the each vertex v_i in K_n, such that $W_{i,j} = v_i + C_{r,i,j}$; $1 \le i \le n$ and $1 \le j \le k$ (A wheel graph $W_{i,j}$ is obtained from a cycle $C_{r,i,j}$ by adding new vertex v_i and joining it to all the r vertices of the cycle by an edge. The new edges are called the spokes of the wheel). The deletion of the spokes of the wheel results in the disjoint union of the complete graph K_n and kn independent cycles $C_{r,i,j}$; $1 \le i \le n$ and $1 \le j \le k$. It is noted that these graphs has diameter atmost 3.



Example 1: Cyclic split graph $C_4K_3^2$

Theorem 3.2:
Let G be a cyclic split graph $C_rK_n^k$ then the harmonious chromatic number of G is given by $\chi_h(G) = n + rk$.

Proof: Let v_i be the vertices of the complete graph in $C_rK_n^k$. Since k wheels are attached to each v_i ; $1 \le j \le n$, there are kn cycles of length r in $C_rK_n^k$. Let $V(C_i) = \{u_{(i-1)r+j} : j = 1,2, \dots, r\}, i = 1,2, \dots \dots kn$.

Define $X = v_1, v_2, \dots \dots, v_n$ and $Y = V(C_i) = \{u_{(i-1)r+j} : j = 1,2, \dots, r\}, i = 1,2, \dots \dots kn$. By the definition of split graph X form a clique of n vertices in $C_nK_r^k$ and Y is an kn independent cycles of length r. We note that $X \cap Y = \emptyset$ and Y is an independent set. Therefore Y can be separated into n sets namely, $T_1, T_2, \dots \dots, T_n$, where each T_i is a k wheel graph obtained from a cycle by adding new vertex v_i and joining it to all the r vertices of the cycle by an edge.

Clearly $V(C_rK_n^k) = X \cup \bigcup_{i=1}^{n} T_i$.

Assign proper coloring to the vertices of $C_rK_n^k$ as follows:

Let c_i is the colour assigned to the vertices of $v_i, 1 \le i \le n$ and c_j is the color assigned to the vertices of $T_1, 1 \le j \le kr$.Since there is no edge between the vertices of $T_i's$ the same set of colors are used for $T_2, \dots \dots, T_n$.

To prove the above said coloring is harmonious. Assume that it is not a harmonious coloring. Since the vertices of X and Y receive distinct colors, there exists pairs of vertices receive the same color pair of colors, which is a contradiction to the fact $X \cap Y = \emptyset$. Therefore coloring is harmonious. Therefore $\chi_h(G) = n + rk$.

Theorem 3.3: Let G be a cyclic split graph $C_rK_n^k$ then the harmonious chromatic number of G is given by $\chi_h(G) = |V(G)|$ if and only if G has diameter 2.

Proof : The theorem follows from the fact that each vertex must receive a distinct color as it is atmost distance 2 from all other vertices.

Theorem 3.4: Let G be a cyclic split graph $C_r K_n^k$ then the harmonious chromatic number of G is given by $\chi_h(G) = n + rk$ if and only if G has diameter 3.
Proof : Proof of this theorem follows from theorem 3.2.

Theorem 3.4: Let G be a cyclic split graph $C_r K_n^k$ then G is diametrically uniform if and only if G has diameter 1.
Proof : The theorem follows from the definition of diametrically uniform and the fact that in a cyclic split graph there are nkr vertices with eccentricity $e_0 = 3$ and n vertices with eccentricity $e_1 = 2$.
Observation 3.5: Let G be a cyclic split graph $C_r K_n^k$ then we have the following, for $k > 1$

(i) When both n and r are even G has perfect matching and $\alpha'(G) = \frac{n(kr+1)}{2}$.
(ii) Whenever r is odd G doesnot have a perfect matching and $\alpha'(G) = \frac{nk(r+1)+n}{2}$ with $n(k-1)$ unsaturated vertices.
(iii) When n is odd and r is even G doesnot have perfect matching and $\alpha'(G) = \frac{nkr+(n-1)}{2}$ with one unsaturated vertices.

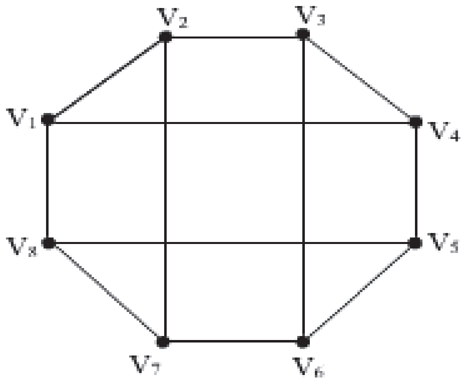
Theorem 3.6: Let G be a cyclic split graph $C_r K_n^k$ then G has perfect matching if and only if G doesnot contain an odd cycle, $k > 1$ as an induced subgraph.
Corollary 3.7: Let G be a cyclic split graph $C_r K_n^k$ then G doesnot have perfect matching if and only if there exist atleast one odd cycle, $k > 1$ as an induced subgraph.

Theorem 3.8: The harmonious chromatic number of central graph of Cyclic split graph is
$\chi_h\{C[C_r K_n^k]\} = \Delta\{C[C_r K_n^k]\} + \Delta\{[C_r K_n^k]\} + 1$ for $k > 1$.
Proof: Let $C_r K_n^k$ be the cyclic split graph with $n(kr+1)$ vertices and $2nkr + \frac{n(n-1)}{2}$ edges. Now by the definition of central graph, subdividing each edge exactly once and joining all the non adjacent vertices, the number of vertices and edges in central graph of the Cyclic split graph $C[C_r K_n^k]$ are $3nkr + \frac{n(n+1)}{2}$ and $\frac{n}{2}[n(k^2r^2 + 2kr + 2) - (2 - 3kr)]$ respectively. Now we assign colors to these vertices as follows. The maximum degree of $C[C_r K_n^k]$ is $n(kr+1) - 1$, hence we need $n(kr+1)$ colors to color these vertices, and it is sufficient to color the remaining $2nkr + \frac{n(n-1)}{2}$ vertices with $n + kr - 1$ colors which is equal to the maximum degree of $C_r K_n^k$. Clearly we need $n(kr+2) + kr - 1$ colors to color the central graph of cyclic split graph. That is $\Delta\{C[C_r K_n^k]\} + \Delta[C_r K_n^k] + 1$ colors.
Hence $\chi_h\{C[C_r K_n^k]\} = \Delta\{C[C_r K_n^k]\} + \Delta\{[C_r K_n^k]\} + 1$ for $k > 1$.
We prove the result by induction on n, k and r.
For n = 3, k = 2 and r = 3 the result is obvious. Assume that the theorem holds for n = n, k =k and r =r then $\chi_h\{C[C_r K_n^k]\} = \Delta\{C[C_r K_n^k]\} + \Delta\{[C_r K_n^k]\} + 1$ for $k > 1$.
Let us verify by induction for n = n + 1, k = k + 1, r = r + 1.

Consider $C[C_r K_{n+1}^k]$. A Cyclic split graph of size n + 1 can be obtained from $C_r K_n^k$ by adding one vertex to the complete graph K_n which in turn increases the number of wheels by k. The vertices and edges are increased by $kr + 1$ and $2kr + n$ respectively. Now by the definition of central graph, the edges $2kr + n$ are subdivided by a new vertex. The total number of vertices s in $C[C_r K_{n+1}^k]$ are $3kr + n + 1$. Removing such vertices in $C[C_r K_{n+1}^k]$, we get a subgraph G' which is nothing but $C[C_r K_n^k]$. Hence by assumption we have $\chi_h\{C[G']\} = \Delta\{C[G']\} + \Delta[G'] + 1$. Now by adding the removal vertices, the maximum degree of $C[C_r K_{n+1}^k]$ is $\Delta\{C[C_r K_n^k]\} + kr + 1$ by the definition of central graph. Moreover $\Delta[C_r K_n^k]$ is same as $\Delta[C_r K_{n+1}^k]$.
Therefore $\chi_h\{C[C_r K_{n+1}^k]\} = \Delta\{C[C_r K_n^k]\} + kr + 1 + \Delta[C_r K_{n+1}^k] + 1$ for $n > 1$
$= n(kr+1) - 1 + kr + 1 + \Delta[C_r K_{n+1}^k] + 1$
$= nkr + n + kr + \Delta[C_r K_{n+1}^k] + 1$
$= (n+1)kr + n + \Delta[C_r K_{n+1}^k] + 1$
$= \Delta\{C[C_r K_{n+1}^k]\} + \Delta[C_r K_{n+1}^k] + 1$
Hence the proof.

Similarly we can prove for k = k + 1 and r = r + 1.

Definition 3.9: Crown graph is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a horizontal edges and is denoted by $H_{n,n}$. The number of vertices in the crown graph $n > 2$ is $2n$ and the number of edges in the crown graph is $n^2 - n$. The maximum and minimum degree of the crown graph is $n - 1$. Moreover the graph has perfect matching and maximum matching number is n. It is a diametrically uniform graph with diameter 3.



Example 2: Crown graph $H_{4,4}$

3.10 Diametrically uniform Graphs: For each vertex u of a graph G, the maximum distance d(u,v) to any other vertex v of G is called its eccentricity and is denoted by ecc(u). In graph G, the maximum value of eccentricity of vertices of G is called the diameter of G and is denoted by λ and the minimum value of eccentricity of vertices of G is called the radius of and it is denoted by ρ. The set of vertices of a graph G with eccentricity equal to the radius ρ is called the center of G and is denoted by Z(G).Let G be a graph with diameter λ. A vertex v of G is said to be diametrically opposite to a vertex u of G, if $d_G(u, v) = \lambda$ (These pairs of vertices are also called diameter). A graph G is said to be diametrically uniform graph if every vertex of g has at least one diametrically opposite vertex. The set of diametrically opposite vertices of x in G is denoted by D(x).

Authors Paul et.al observed that BFS(u) of a graph G denotes a breadth first search (bfs) tree of G rooted at the vertex at the vertex u[5]. They have observed a close relationship between breadth first search (bfs) trees and these classes of graphs. Given two vertices u and v of a vertex transitive graph, for every bfs tree BFS(u), there exist a bfs tree BFS(v) such that BFS(U) and BFS(u) are isomorphic. Moreover, any two bfs trees of a diametrically uniform graph are of equal height.

Proposition 3.11: The crown graph $H_{n,n}$ diametrically uniform. Moreover, for any vertex x in $H_{n,n}$, the set $D(x)$ of diametrically opposite vertices of x satisfies $|D(x)| = 1\ for\ n > 2$.
The proof of this Proposition follows by considering the breath first search tree rooted at the vertex x of the graph.

Theorem 3.12: Let G be a crown graph $H_{n,n}$ then the harmonious chromatic number of G is given by $\chi_h(G) = 2(\alpha'(G)) - |D(x)|$
Observation 3.13
Let G be a crown graph $H_{n,n}$, $n > 2$. Then

1. For every edge (x, y) of G, $D(x) \cup D(y)$ is connected
2. For every vertex x of $G, D(x)$ is a singleton.

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