FIXED POINT RESULTS IN CONE 2-METRIC SPACE

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Abstract: In the present paper the fixed point theorem in cone-2 metric spaces for rational expressions. The result generalized previous known results.

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Introduction & Preliminaries: There are many generalizations of metric spaces: Menger space, Fuzzy metric space, Generalized metric spaces, Cone metric space, K- metric space and K-normed space, Rectangular metric and Rectangular cone metric space.

In 2007, Huang and Zhang [52] introduced cone metric spaces being unaware that they already existed under the name K-metric, and K- normed spaces that were introduced and used in the middle of the 20th century . In both cases the set R of real numbers was replaced by an ordered Banach space E. However, Huang and Zhang [52] went further and defined the convergence via interior points of the cone by which the order in E is defined. This approach allows the investigation of cone spaces in the case when the cone is not necessarily normal. And yet, they continued with results concerned with the normal cones only. One of the main results is the following Banach Contraction Principle in the setting of normal cone spaces:

Theorem A: Let (X, d) be a complete cone metric space over a normal solid cone. Suppose that a mapping $T : X \rightarrow X$ satisfies the contractive condition

$$d(Tx, Ty) \leq \lambda\, d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in X and, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

In this present paper we prove some more results in cone 2- metric space satisfying symmetric rational contractive conditions. This chapter is divided into three sections in which we proves different fixed point and common fixed point theorems satisfying symmetric rational contractive conditions. To prove of our main results first we give some known results which are used in this chapter.

Definition B: Let X be a Real Banach Space and P a subset of X, P is called a cone if P satisfy followings conditions;

a. P is closed , nonempty and $P \neq 0$
b. $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b
c. $P \cap (-P) = \{0\}$

Given a cone $P \subset X$, we define a partial ordering $\leq$ on X with respect to P by $y - x \in P$. we shall write $x << y$ if $(y - x) \in \operatorname{int} P$, denoted by $\|.\|$ the norm on X . The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in X$

$$0 \leq x \leq y \text{ implies that } \|x\| \leq k\,\|y\| \qquad (A)$$

The least positive number k satisfying the above condition (A) is called the normal constant of P.
The cone P is called regular if every increasing sequence which is bounded from the above is convergent, that is if $\{x_n\}_{n\geq 1}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq y$$

for some $y \in X$, then there is $x \in X \lim_{n\to\infty} \|x_n - x\| = 0$.
The cone P is regular iff every decreasing sequence which is bounded from below is convergent

Definition C: let X be a nonempty set and X is a real Banach Space, d is a mapping from X into itself a> 0, such that, d satisfying following conditions,

$d_1 : d(x,y,a) \geq 0 \quad \forall\, x,y \in X$
d_2: $d(x,y,a) = 0$ iff $x = y$
d_3: $d(x,y,a) = d(y,x,a)$
d_4: $d(x,y,a) \leq d(x,z,a) + d(z,y,a)$
Then d is called a cone 2- metric on X and (X,d) is called cone metric space.

Definition E: Let A and S be two mapping of a cone 2- metric space (X,d) then it is said to be compatible if, $\lim_{n\to\infty} d(ASx_n, SAx_n, a) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = t \text{ and } \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Definition F: Let A and S be two self mapping of a cone 2- metric space (X,d) then it is said to be Weakly compatible, if they commute at coincidence point , that is $Ax = Sx$ implies that,
$ASx = SAx$ for $x \in X$.
It is easy to see that compatible mapping commute at their coincidence points. It is note that a compatible maps are weakly compatible but converges need not be true.

Main Result: The aim of this paper is to prove some common fixed point results in cone metric space satisfying symmetric rational contractive conditions. Now we give our first main result of this section which as follows

Theorem 3.1: Let (X,d) be a complete 2- cone metric space and P a normal cone with normal constant k. Suppose that the mapping T, from X into itself satisfy the condition,

$$d(Tx,Ty,a) \leq \alpha\, d(x,y,a) + \beta \frac{d^2(x,Tx,a) + .d^2(y,Ty,a)}{1 + d(x,Tx,a) + d(y,Ty,a)} + \gamma \frac{d^2(x,Ty,a) + d^2(y,Tx,a)}{1 + d(x,Ty,a) + d(y,Tx,a)} \quad \text{-- (3.1a)}$$

For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta, \eta \geq 0$ such that $0 \leq \alpha + 2\beta + 2\gamma < 1$. Then T has fixed point in X. If $0 \leq \alpha + 2\gamma < 1$ then T has unique fixed point in X.

Proof: For any arbitrary x_0 in X, we choose $x_1, x_2 \in X$ such that,
$Tx_0 = x_1$ and $Tx_1 = x_2$.
In general we can define a sequence of elements of X such that,
$x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$
Now,
$d(x_{2n+1}, x_{2n+2}a) = d(Tx_{2n}, Tx_{2n+1,a})$
From 3.1(a)

$$d(Tx_{2n}, Tx_{2n+1}, a) \leq \alpha\, d(x_{2n}, x_{2n+1}, a) + \beta \frac{d^2(x_{2n}, Tx_{2n}, a) + d^2(x_{2n+1} Tx_{2n+1,a})}{1 + d(x_{2n}, Tx_{2n}, a) + d(x_{2n+1} Tx_{2n+1}, a)} + \gamma \frac{d^2(x_{2n}, Tx_{2n+1}, a) + d^2(x_{2n+1}, Tx_{2n}, a)}{1 + d(x_{2n}, Tx_{2n+1}, a) + d(x_{2n+1}, Tx_{2n,a})}$$

$$d(x_{2n+1}, x_{2n+2}, a) \leq \alpha\, d(x_{2n}, x_{2n+1,}a)$$

$$+\beta \frac{d^2(x_{2n}, x_{2n+1}, a) + d^2(x_{2n+1} x_{2n+2}, a)}{1 + d(x_{2n}, x_{2n+1}) + d(x_{2n+1} x_{2n+2})} + \gamma \frac{d^2(x_{2n}, x_{2n+2}, a) + d^2(x_{2n+1}, x_{2n+1,a})}{1 + d(x_{2n}, x_{2n+2}, a) + d(x_{2n+1}, x_{2n+1}, a)}$$

$$d(x_{2n+1}, x_{2n+2}, a) \leq \alpha\, d(x_{2n}, x_{2n+1}, a) + \beta\, [d(x_{2n}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n+2}, a)] + \gamma \,.\, d(x_{2n}, x_{2n+2}, a)$$

By using triangle inequality, we get,

$$d(x_{2n+1}, x_{2n+2}, a) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma-\delta}\right] d\big(x_{2n}, x_{2n+1,} a\big)$$

Similarly we can show that,

$$d(x_{2n}, x_{2n+1}, a) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma-\delta}\right] d(x_{2n-1}, x_{2n}, a)$$

In general we can write,

$$d(x_{2n+1}, x_{2n+2}, a) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma-\delta}\right]^{2n+1} d(x_0, x_1, a)$$

On taking $\left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma-\delta}\right] = \theta$

$$d\big(x_{2n+1}, x_{2n+2,} a\big) \leq \theta^{2n+1} d(x_0, x_1, a)$$

For $n \leq m$, we have

$$d(x_{2n}, x_{2m}, a) \leq d(x_{2n}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n+2}, a) + \cdots \ldots \ldots \ldots \ldots .. + d(x_{2m-1}, x_{2m}, a)$$

$$d(x_{2n}, x_{2m}, a) \leq \{\theta^n + \theta^{n+1} + \theta^{n+2} + \cdots \ldots \ldots .. + \theta^m\} d(x_0, x_1, a)$$

$$d(x_{2n}, x_{2m}, a) \leq \frac{\theta^n}{1-\theta} d(x_0, x_1, a)$$

$$\|d(x_{2n}, x_{2m}, a)\| \leq \frac{\theta^n}{1-\theta} k \| d(x_0, x_1, a) \|$$

as $n \to \infty$

$$\lim_{n\to\infty} \|d(x_{2n}, x_{2m}, a)\| \to 0$$

In this way

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) \to 0, \text{ as } n \to \infty$$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$.
Hence (X, d) is complete cone metric space.
Thus $x_n \to u$ as $n \to \infty$, $Tx_{2n} \to u$ and $T_{2n+1} \to u$ as $n \to \infty$
u is fixed point of T in X.

Uniqueness: Let us assume that, v is another fixed point of T in X different from v. then, $Tu = u$ and $Tv = v$

$$d(u, v, a) = d(Tu, Tv, a)$$

From 3.1 (a)

$$d(Tu, Tv, a) \leq \alpha\, d(u, v, a) + \beta \frac{d^2(u,Tu,a)+d^2(v,Tv,a)}{1+d(u,Tu,a)+d(v,Tv,a)} + \gamma \frac{d^2(u,Tv,a)+ d^2(v,Tu,a)}{1+d(u,Tv,a)+ d(v,Tu,a)}$$

$$d(Tu, Tv, a) \leq (\alpha + 2\gamma).\ d(u, v, a)$$

Which contradiction so u is unique fixed point of T in X.

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