VERTEX-TO-CLIQUE CONCEPTS IN GRAPHS

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Abstract: In this paper, the vertex-to-clique $u - C$ path, the vertex-to-clique $u - C$ monophonic path and the vertex-to-clique $u - C$ triangle free path are studied. Correspondingly, the vertex-to-clique distance $d(u, C)$, the vertex-to-clique monophonic distance $d_m(u, C)$, the vertex-to-clique triangle free detour distance $D_{\Delta f}(u, C)$ and the vertex-to-clique detour distance $D(u, C)$ are studied. Several results concerning these four distances are presented. It is shown that the vertex-to-clique radius $rad_1(G)$, the vertex-to-clique monophonic radius $rad_1{}^*(G)$, the vertex-to-clique triangle free detour radius $rad_{\Delta f_1}(G)$ and the vertex-to-clique detour radius $rad_{D_1}(G)$ are realizable in a connected graph G. Also it is shown that the vertex-to-clique diameter $diam_1(G)$, the vertex-to-clique monophonic diameter $diam_1{}^*(G)$, the vertex-to-clique triangle free detour diameter $diam_{\Delta f_1}(G)$ and the vertex-to-clique detour diameter $diam_{D_1}(G)$ are realizable in a connected graph G.

Keywords: Distance, Detour Distance, Monophonic Distance, Triangle Free Detour Distance.

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Introduction: The centrality in graphs arise in the context of selection of a site at which locate a facility in a graph. Many problems of finding the best site for a facility in a graph or network are in minimax location problems. For any two vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. Ordinarily, when we wish to proceed from a point A to a point B we take a route which involves the least distance. We have all been faced with detour signs which require us to take a route from A to B that involves a greater distance. In any such detour route from A to B we assume that there is no possible shortcut along the route, for otherwise this should have been part of the route initially. When one is driving along such a detour, it sometimes seems that we are using the longest route possible from A to B. For a non empty set S of vertices of G, the subgraph $< S >$ of G induced by S has S as its vertex set while an edge of G belongs to $< S >$ if it joins two vertices of S. If P is a $u - v$ path of length $d(u, v)$, then the subgraph $< V(P) >$ induced by the vertices of P is P itself. This observation suggested to introduce the detour distance in graphs by Chartrand, Johns and Tian [2] as follows. For any two vertices u and v in a connected graph G, the detour distance $d^*(u, v)$ is the length of a longest induced $u - v$ path in G. That is, a longest $u - v$ path P for which $< V(P) >= P$. An induced $u - v$ path of length $d^*(u, v)$ is called a detour path. This detour path contains no chords between any two non-adjacent vertices of P. In 2011, Santhakumaran and Titus [8] rebuilt this detour distance as monophonic distance in graphs. That is, the monophonic distance from u to v is defined as the length of a longest $u - v$ chordless path in G, both are equal in G. Since there is another detour distance D is studied by Chartrand et. al, the detour distance d^* is renamed as monophonic distance d^* in G for our convenience. It is observed that the route from A to B involves the longest distance other than the monophonic distance, which also have been faced with detour signs. In 2005, Chartrand, Escuadro and Zhang [1] studied the concepts of another detour distance in graphs. For any two vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. Also the route from A to B involves one more distance other than the monophonic distance and the detour distance, which also have been faced with detour signs. For example if one wants to design a security based communication network, the triangle free detour distance play a vital role. These ideas have interesting applications in channel assignment problem in radio technologies. In the case of designing the channel for a communication network, although maximum number of vertices are covered by the network when considering paths, some of the edges may be left out. This drawback is rectified in the case of triangle free detour distance. Keerthi Asir and Athisayanathan [4] introduced and studied the concepts of triangle free detour distance in graphs. For any two vertices u and v in G, a

$u - v$ path P is a $u - v$ triangle free path if P contains no triangles. The triangle free detour distance $D_{\Delta f}(u,v)$ from u to v is defined as the length of a longest $u - v$ triangle free path in G. A clique C of a graph G is a maximal complete subgraph and we denote it by its vertices. Terms not defined here may be found in [3,4,7]. In a social network a clique represents a group of individuals having a common interest. Thus the centrality with respect to cliques have interesting application in social networks. For example when a railway line, pipe line, or highway and dam, lake or pond is constructed, the distance between the respective structure and each of the communities to be served is to be minimized/minimized. These motivated us to study the vertex-to-clique concepts in graphs. Throughout this paper G denotes a connected undirected simple graph with at least two vertices.

Main Results:

Definition 2.1 Let u be a vertex and C a clique in a connected graph G. A vertex-to-clique $u - C$ path P is a $u - v$ path, where v is a vertex in C such that P contains no vertices of C other than v. The vertex-to-clique distance $d(u,C)$ is the length of a shortest $u - C$ path in G. The vertex-to-clique eccentricity $e_1(u)$ of u in G is defined as $e_1(u) = max\{d(u,C) : C \in \zeta\}$, where ζ is the set of all cliques in G. The vertex-to-clique radius of G is defined as, $rad_1(G) = min\{e_1(v) : v \in V\}$ and the vertex-to-clique diameter of G is defined as, $diam_1(G) = max\{e_1(v) : v \in V\}$. A vertex v in G is called a vertex-to-clique central vertex if $e_1(v) = rad_1(G)$ and the vertex-to-clique center of G is defined as, $C_1(G) = Cen_1(G) = < \{v \in V : e_1(v) = rad_1(G)\} >$. A vertex v in G is called a vertex-to-clique peripheral vertex if $e_1(v) = diam_1(G)$ and the vertex-to-clique periphery of G is defined as, $P_1(G) = Per_1(G) =< \{v \in V: e_1(v) = diam_1(G)\} >$. The vertex-to-clique detour distance $D(u,C)$ is the length of a longest $u - C$ path in G and its parameters are defined as above in [5]. The vertex-to-clique monophonic distance, $d^*(u,C)$ is the length of a longest $u - C$ path P for which $< V(P) > = P$ and its parameters are defined as above in [6]. More over the vertex-to-clique $u - C$ path P is said to be a vertex-to-clique $u - C$ triangle free path if no three vertices of P induce a cycle C_3 in G. The vertex-to-clique triangle free detour distance, $D_{\Delta f}(u,C)$ is the length of a longest $u - C$ triangle free path in G and its parameters are defined as above.

Example 2.2 For the vertex u and the clique $C = \{v,w\}$, in Fig 2.1, $Q_1: u,w$, $Q_2: u,z,r,v$, $Q_3: u,t,s,x,z,r,v$ and $Q_4: u,t,s,x,y,z,r,v$ are the $u - C$ paths and no other $u - C$ paths exist. Thus $d(u,C) = 1$, $d^*(u,C) = 3$, $D_{\Delta f}(u,C) = 6$ and $D(u,C) = 7$. It is clear that all the above distances are distinct.

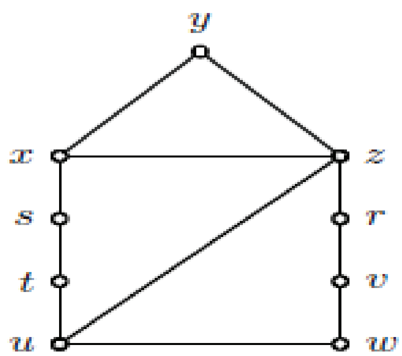

Fig 2.1: G

We can easily verify that these distances are not metric in a vertex set of a connected graph G. It is observed that the length of a $u - C$ path between a vertex u and a clique C in a graph G of order n is at most $n - 2$, the following theorem gives the relations among these distances in any graph G.

Theorem 2.3. For any vertex u and a clique C in a connected graph G, $0 \leq d(u,C) \leq d^*(u,C) \leq D_{\Delta f}(u,C) \leq D(u,C) \leq n - 2$.

Proof: By definition $d(u,C) \leq d^*(u,C)$ and $d^*(u,C) \leq D(u,C)$. It is enough to show that $d^*(u,C) \leq D_{\Delta f}(u,C)$ and $D_{\Delta f}(u,C) \leq D(u,C)$. Let P be any longest $u-C$ path in G. Suppose that P does not induces a chord in G, then $d^*(u,C) = D_{\Delta f}(u,C) = D(u,C)$. Suppose that P induce a chord.
Case 1. If P induce a cycle C_3 in G, then $d^*(u,C) = D_{\Delta f}(u,C) < D(u,C)$.
Case 2. If P does not induce a cycle C_3 in G, then $d^*(u,C) < D_{\Delta f}(u,C) = D(u,C)$.

Example 2.4. For the graph G given in Fig. 2.2, the vertex-to-clique eccentricity $e_1(v)$, the vertex-to-clique monophonic eccentricity $e_1{}^*(v)$, the vertex-to-clique triangle free detour eccentricity $e_{\Delta f_1}(v)$, and vertex-to-clique detour eccentricity $e_{D_1}(v)$ of all the vertices of G are given in Table 1. The vertex-to-clique radius $rad_1(G) = 2$, the vertex-to-clique diameter $diam_1(G) = 4$, the vertex-to-clique monophonic radius $rad^*{}_1(G) = 4$, the vertex-to-clique monophonic diameter $diam^*{}_1(G) = 6$, the vertex-to-clique triangle free detour radius $rad_{\Delta f_1}(G) = 4$, the vertex-to-clique triangle free detour diameter $diam_{\Delta f_1}(G) = 6$, the vertex-to-clique detour radius $rad_{D_1}(G) = 5$, and the vertex-to-clique detour diameter $rad_{D_1}(G) = 10$.

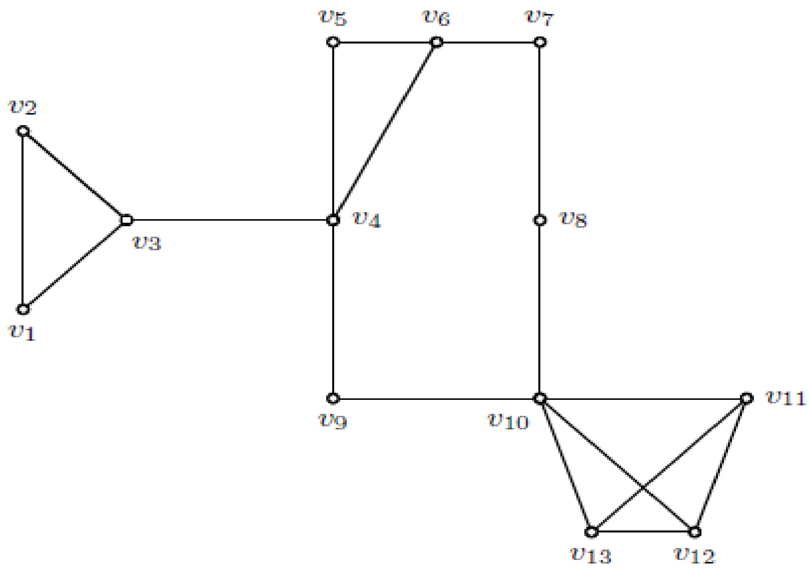

Fig. 2.2: G

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$e_1(v)$	4	4	3	2	3	3	3
$e_1{}^*(v)$	6	6	5	4	5	4	5
$e_{\Delta f_1}(v)$	6	6	5	4	5	6	5
$e_{D_1}(v)$	8	8	6	5	7	6	5

v	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
$e_1(v)$	4	2	3	4	4	4
$e_1{}^*(v)$	4	4	5	6	6	6
$e_{\Delta f_1}(v)$	5	6	5	6	6	6
$e_{D_1}(v)$	6	7	6	10	10	10

Table 1.

Further the vertex-to-clique center $C_1(G) =< \{v_4, v_9\} >$, the vertex-to-clique periphery $P_1(G) = < \{v_1, v_2, v_8, v_{11}, v_{12}, v_{13}\} >$, the vertex-to-clique monophonic center $C_1{}^*(G) = < \{v_4, v_6, v_8, v_9\} >$, the

vertex-to-clique monophonic periphery $P_1^*(G) = \ <\{v_1, v_2, v_{11}, v_{12}, v_{13}\}>$, the vertex-to-clique triangle free detour center $C_{\Delta f_1}(G) = <\{v_4\}>$, and the vertex-to-clique triangle free detour periphery $P_{\Delta f_1}(G) = <\{v_1, v_2, v_6, v_9, v_{11}, v_{12}, v_{13}\}>$, the vertex-to-clique detour center $C_{D_1}(G) =<\{v_4, v_7\}>$ and the vertex-to-clique detour periphery $P_{D_1}(G) =<\{v_{11}, v_{12}, v_{13}\}>$. It is clear that all the parameters defined above are distinct.
The following theorem is a consequence of Theorem 2.3.

Theorem 2.5: Let G be a connected graph of order n. Then
(i) $0 \leq e_1(v) \leq e_1^*(v) \leq e_{\Delta f_1}(v) \leq e_{D_1}(v) \leq \ n-2$.
(ii) $0 \leq rad_1(G) \leq rad_1^*(G) \leq rad_{\Delta f_1}(G) \leq rad_{D_1}(v) \leq n-2$.
(iii) $0 \leq diam_1(G) \leq diam_1^*(G) \leq diam_{\Delta f_1}(G) \leq diam_{D_1}(G) \leq n-2$.
The vertex-to-clique radius $rad_1(G)$, the vertex-to-clique monophonic radius $rad_1^*(G)$, the vertex-to-clique triangle free detour radius $rad_{\Delta f_1}(G)$, and the vertex-to-clique detour radius $rad_{D_1}(G)$ of some standard graphs are given in Table 2.

Graph G	K_n	P_n	$C_n(n \geq 4)$
$rad_1(G)$	0	$\lfloor \frac{n-2}{2} \rfloor$	$\lfloor \frac{n-1}{2} \rfloor$
$rad_1^*(G)$	0	$\lfloor \frac{n-2}{2} \rfloor$	$n-2$
$rad_{\Delta f_1}(G)$	0	$\lfloor \frac{n-2}{2} \rfloor$	$n-2$
$rad_{D_1}(G)$	0	$\lfloor \frac{n-2}{2} \rfloor$	$n-2$

Graph G	$W_n(n \geq 5)$	$K_{n,m}(2 \leq n \leq m)$
$rad_1(G)$	0	1
$rad_1^*(G)$	0	2
$rad_{\Delta f_1}(G)$	0	$2(n-1)$
$rad_{D_1}(G)$	0	$2(n-1)$

Table 2.

Also the vertex-to-clique diameter $diam_1(G)$, the vertex-to-clique monophonic diameter $diam_1^*(G)$, the vertex-to-clique triangle free detour diameter $diam_{\Delta f_1}(G)$, and the vertex-to-clique detour diameter $diam_{D_1}(G)$ of some standard graphs are given in Table 3. The results given in Tables 2 and 3 has the desired property for the upper and lower bounds given in Theorem 2.5. It is observed that either $e_1(v) \ = \ e(v)$ or $e(v) - 1$ for a vertex v in G and there is no vertex v in G such that $e_{D_1}(v) \ = \ e_D(v)$. Also we can easily verify that for some vertex v in G, $e_1^*(v) \ = \ e^*(v)$ and $e_{\Delta f_1}(v) \ = e_{\Delta f}(v)$.

Graph G	K_n	P_n	$C_n(n \geq 4)$
$diam_1(G)$	0	$n-2$	$\lfloor \frac{n-1}{2} \rfloor$
$diam_1^*(G)$	0	$n-2$	$n-2$
$diam_{\Delta f_1}(G)$	0	$n-2$	$n-2$
$diam_{D_1}(G)$	0	$n-2$	$n-2$

Graph G	$W_n (n \geq 5)$	$K_{n,m} (2 \leq n \leq m)$
$diam_1(G)$	1	1
$diam_1{}^*(G)$	$n-3$	2
$diam_{\triangle f_1}(G)$	$n-3$	$\begin{cases} 2(n-1) & \text{if } n=m \\ 2n-1 & \text{if } n>m \end{cases}$
$diam_{D_1}(G)$	$n-3$	$\begin{cases} 2(n-1) & \text{if } n=m \\ 2n-1 & \text{if } n>m \end{cases}$

Table 3.

For a graph G, $rad_1(G) \leq diam_1(G) \leq 2rad_1(G) + 1$. The upper inequality does not hold for the vertex-to-clique monophonic distance, vertex-to-clique triangle free detour distance, and vertex-to-clique detour distance in G. For the wheel graph $G = W_n (n \geq 5)$, $diam_1{}^*(G) > 2rad_1{}^*(G) + 1$, $diam_{\triangle f_1}(G) > 2rad_{\triangle f_1}(G) + 1$, and $diam_{D_1}(G) > 2rad_{D_1}(G) + 1$. Next we find the answer for the question, "Which graphs can be the vertex-to-clique center of some graph?". The answer is every graph. Also the vertex-to-clique center of every graph G lies in a single block of G.

Next We Give Some Open Problems:

Problem 2.6: Is every graph a vertex-to-clique monophonic center of some connected graph?

Problem 2.7. Is every graph a vertex-to-clique triangle free detour center of some connected graph?

Problem 2.8. Is every graph a vertex-to-clique detour center of some connected graph?

Problem 2.9. Does the vertex-to-clique monophonic center of every graph G lie in a single block of G?.

Problem 2.10. Does the vertex-to-clique triangle free detour center of every graph G lie in a single block of G?.

Problem 2.11. Does the vertex-to-clique detour center of every graph G lie in a single block of G?.

Realization Results: Now we have a realization theorem for the vertex-to-clique radius and the vertex-to-clique diameter of some connected graph.

Theorem 3.1: For each pair a, b of positive integers with $a \leq b \leq 2a + 1$, there exists a connected graph G with $rad_1(G) = a$ and $diam_1(G) = b$.
Proof. We prove this theorem by the two cases as follows:
Case 1. $a = b$. Let $G = C_{2a+1} : u_1, u_2, \ldots, u_a, v_a, \ldots, v_2, v_1, w$ be a cycle of order $2a + 1$. It is easy to verify that $e_1(w) = e_1(u_i) = e_1(v_i) = a$ for $1 \leq i \leq a$. Thus $rad_1(G) = a$ and $diam_1(G) = b$ as $a = b$.
Case 2. $a < b \leq 2a + 1$. We have the following two sub cases:
Subcase 1 of Case 2. $a = 1$. Then $b = 2$ or 3. Now we have the following two cases:
Case (i) of Subcase 1. If $b = 2$, then $G = P_4$ is the desired graph.
Case (ii) of Subcase 1. If $b = 3$, then $G = P_5$ is the desired graph.
Subcase 2 of Case 1. $a \geq 2$. Let $C_{2a+1} : u_1, u_2, \ldots, u_a, v_a, \ldots, v_2, v_1, w$ be a cycle of order $2a + 1$ and $P_{b-a+1}: w_1, w_2, \ldots, w_{b-a+1}$ be a path of order $b - a + 1$. We construct the graph G of order $b + a + 2$ by identifying the vertex w of C_{2a+1} with w_1 of P_{b-a+1} as shown in the Fig 3.1. It is easy to verify that

$e_1(w) = a$

$e_1(u_i)$ =max$\{\left\lfloor\frac{2a+1}{2}\right\rfloor, b-a+i-1\}$ for $1 \le i \le a$
$e_1(v_i)$ =max$\{\left\lfloor\frac{2a+1}{2}\right\rfloor, b-a+i-1\}$ for $1 \le i \le a$
$e_1(w_i) = a+i-1$ for $1 \le i \le b-a+1$

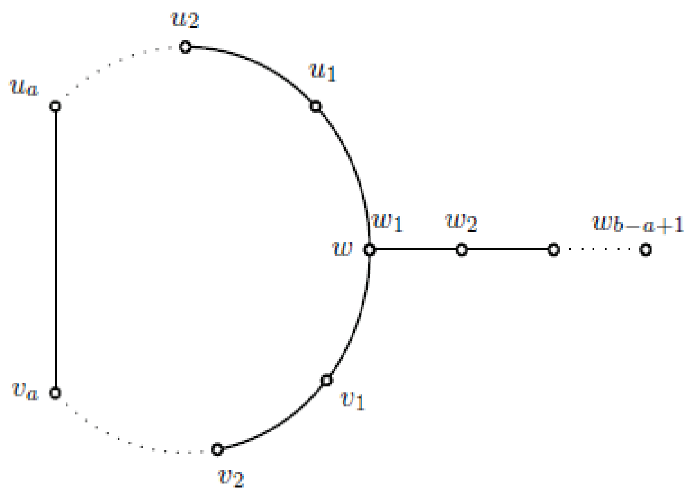

Fig. 3.1: G

It is easy to verify that there is no vertex x in G with $e_1(x) < a$ and there is no vertex y in G with $e_1(y) > b$. Thus $rad_1(G) = a$ and $diam_1(G) = b$ as $a < b$.

Problem 3.2. For each pair a, b of positive integers with $2 \le a \le b$, does there exist a connected graph G with $rad_1{}^*(G) = a$ and $diam_1{}^*(G) = b$?.

Problem 3.3. For each pair a, b of positive integers with $2 \le a \le b$, does there exist a connected graph G with $rad_{\Delta f_1}(G) = a$ and $diam_{\Delta f_1}(G) = b$?.

Problem 3.4. For each pair a, b of positive integers with $2 \le a \le b$, does there exist a connected graph G with $rad_D(G) = a$ and $diam_D(G) = b$?.

Now we have a realization theorem for the vertex-to-clique radius, the vertex-to-clique monophonic radius, the vertex-to-clique triangle free detour radius and the vertex-to-clique detour radius as follows.

Theorem 3.5. For any four positive integers a, b, c and d with $5 \le a \le b \le c \le d$, there exists a connected graph G such that $rad_1(G) = a$, $rad_1{}^*(G) = b$, $rad_{\Delta f_1}(G) = c$ and $rad_{D_1}(G) = d$.
Proof: ***Case 1.*** $a = b = c = d$. Let $G = P_{2a+3}: u_1, u_2, \ldots, u_{2a+3}$ be a path of order $2a+3$. Then $e_1(u_i) = e_1{}^*(u_i) = e_{\Delta f_1}(u_i) = e_{D_1}(u_i) = a$ if $i = a+2$. It is easy to verify that there is no vertex x in G with $e_1(x) < a$, $e_1{}^*(x) < b$, $e_{\Delta f_1}(x) < c$ and $e_{D_1}(x) < d$. Thus $rad_1(G) = a$, $rad_1{}^*(G) = b$, $rad_{\Delta f_1}(G) = c$ and $rad_{D_1}(G) = d$ as $a = b = c = d$.
Case 2. $5 \le a \le b \le c < d$. Let $F_1 : u_1, u_2, \ldots, u_{a+1}$ and $F_2 : v_1, v_2, \ldots, v_{a+1}$ be two copies of the path P_{a+1} of order $a+1$. Let $F_3 : w_1, w_2, \ldots, w_{b-a+3}$ and $F_4 : z_1, z_2, \ldots, z_{b-a+3}$ be two copies of the path P_{b-a+3} of order $b-a+3$. Let $F_5 : s_1, s_2, \ldots, s_{c-b+2}$ and $F_6 : t_1, t_2, \ldots, t_{c-b+2}$ be two copies of the path P_{c-b+2} of order $c-b+2$. Let $F_7 = K_{d-c+1}$ be the complete graph of order $d-c+1$ with $V(F_7) = \{x_1, x_2, \ldots, x_{d-c+1}\}$. Choose a vertex $x = u_{a-2}$ in F_1 and $y = v_{a-2}$ in F_2. We construct the graph G as follows: (i) identify the vertices u_1 in F_1; w_1 in F_3 and x_1 in F_7, also identify the vertices v_1 in F_2; z_1 in F_4 and x_{d-c+1} in F_7, (ii) identify the vertices u_3 in F_1 and w_{b-a+3} in F_3, also and identify the vertices z_{b-a+3} in F_4 and v_3 in F_2, (iii) join each vertex $w_i (2 \le i \le b-a+2)$ in F_3 and u_2 in F_1 and join each vertex $z_i (2 \le i \le b-a+2)$ in F_4 and v_2 in F_2, (iv) identify the vertices u_a in F_1 and s_1 in F_5 and identify the vertices va in F2 and t1 in F6, (v) join each vertex $s_i (1 \le i \le c-b+2)$ (not each i, necessarily i is even)

in F_5 with the vertex x and join each vertex $t_i(1 \le i \le c-b+2)$ (not each i, necessarily i is even) in F_6 with the vertex y. It is easy to verify that $e_1(x_i) = a$, $e_1{}^*(x_i) = b$, $e_{\Delta f_1}(x_i) = c$, $e_{D_1}(x_i) = d$ if $1 \le i \le d - c + 1$. It is easy to verify that there is no vertex x in G with $e_1(x) < a$, $e_1{}^*(x) < b$, $e_{\Delta f_1}(x) < c$ and $e_{D_1}(x) < d$. Thus $rad_1(G) = a$, $rad_1{}^*(G) = b$, $rad_{\Delta f_1}(G) = c$ and $rad_{D_1}(G) = d$ as $a \le b \le c < d$.

Case 3. $5 \le a \le b \le c = d$. Let $E_1 : v_1, v_2, \ldots, v_{2a+3}$ be a path of order $2a+3$. Let $E_2 : u_1, u_2, \ldots, u_{b-a+3}$ and $E_3 : w_1, w_2, \ldots, w_{b-a+3}$ be two copies of the path P_{b-a+3} of order $b - a + 3$. Let $F_1 : s_1, s_2, \ldots, s_{c-b+2}$ and $F_2 : t_1, t_2, \ldots, t_{c-b+2}$ be two copies of the path P_{c-b+2} of order $c - b + 2$. Let $E_i : z_i(4 \le i \le 2(b-a)+3)$ be $2(b-a)$ copies of K_1. Choose a vertex $x = v_4$ and $y = v_{2a}$. We construct the graph G as follows: (i) identify the vertices v_{a+2} in E_1; u_1 in E_2 and w_1 in E_3, (ii) identify the vertices v_a in E_1 and u_{b-a+3} in E_2 and identify the vertices v_{a+4} in E_1 and w_{b-a+3} in E_3 (iii) join each $E_i(4 \le i \le b-a+3)$ with v_{a+2} in E_1 and u_{i-1} in E_2 and join each $E_i(b-a+4 \le i \le 2(b-a)+3)$ with v_{a+2} in E_1 and $w_{i-b+a-1}$ in E_3, (v) identify the vertices t_1 in F_2 and v_{2a+2} in E_1; and identify the vertices s_1 in F_1 and v_2 in E_2; (vi) join each vertex $s_i(1 \le i \le c-b+2)$ (not each i, necessarily i is even) in F_1 with the vertex x; and join each vertex $t_i(1 \le i \le c-b+2)$ (not each i, necessarily i is even) in F_2 with the vertex y. It is easy to verify that $e_1(v_{a+2}) = a$, $e_1{}^*(v_{a+2}) = b$, $e_{\Delta f_1}(v_{a+2}) = c$, $e_{D_1}(v_{a+2}) = d$. It is easy to verify that there is no vertex x in G with $e_1(x) < a$, $e_1{}^*(x) < b$, $e_{\Delta f_1}(x) < c$ and $e_{D_1}(x) < d$. Thus $rad_1(G) = a$, $rad_1{}^*(G) = b$, $rad_{\Delta f_1}(G) = c$ and $rad_{D_1}(G) = d$ as $a \le b \le c = d$.

Now we have a realization theorem for the vertex-to-clique diameter, the vertex-to-clique monophonic diameter, the vertex-to-clique triangle free detour diameter and the vertex-to-clique detour diameter as follows.

Theorem 3.6: For any four positive integers a, b, c and d with $5 \le a \le b \le c \le d$, there exists a connected graph G such that $diam_1(G) = a$, $diam_1{}^*(G) = b$, $diam_{\Delta f_1}(G) = c$ and $diam_{D_1}(G) = d$.

Proof: ***Case 1.*** $a = b = c = d$. Let $G = P_{a+2} : u_1, u_2, \ldots, u_{a+2}$ be a path of order $a+2$. Then $e_1(u_1) = e_1{}^*(u_1) = e_{\Delta f_1}(u_1) = e_{D_1}(u_1) = a$. It is easy to verify that there is no vertex x in G with $e_1(x) > a$, $e_1{}^*(x) > b$, $e_{\Delta f_1}(x) > c$ and $e_{D_1}(x) > d$. Thus $diam_1(G) = a$, $diam_1{}^*(G) = b$, $diam_{\Delta f_1}(G) = c$ and $diam_{D_1}(G) = d$ as $a = b = c = d$.

Case 2. $5 \le a \le b \le c < d$. Let $F_1 : u_1, u_2, \ldots, u_{a+1}$ be a path of order $a+1$. Let $F_2 : w_1, w_2, \ldots, w_{b-a+3}$ be a path of order $b-a+3$. Let $F_3 : s_1, s_2, \ldots, s_{c-b+2}$ be a path of order $c - b + 2$. Let $F_4 = K_{d-c+1}$ the complete graph of order $c - b + 1$ with $V(F_4) = \{x_1, x_2, \ldots, x_{d-c+1}\}$. Choose a vertex $x = u_{a-2}$ in F_1. We construct the graph G as follows: (i) identify the vertices u_1 in F_1; w_1 in F_2 and x_1 in F_4 and identify the vertices u_3 in F_1 and w_{b-a+3} in F_2, (ii) join each vertex $w_i(2 \le i \le b-a+2)$ in F_2 and u_2 in F_1. (iii)identify the vertices s_1 in F_3 and u_a in F_1 and join each vertex $s_i(1 \le i \le c-b+1)$ (not each i, necessarily i is even) in F_3 with the vertex x. It is easy to verify that $e_1(x_i) = a$, $e_1{}^*(x_i) = b$, $e_{\Delta f_1}(x_i) = c$, $e_{D_1}(x_i) = d$ if $2 \le i \le d - c + 1$. Also it is easy to verify that there is no vertex x in G with $e_1(x) > a$, $e_1{}^*(x) > b$, $e_{\Delta f_1}(x) > c$ and $e_{D_1}(x) > d$. Thus $diam_1(G) = a$, $diam_1{}^*(G) = b$, $diam_{\Delta f_1}(G) = c$ and $diam_{D_1}(G) = d$ as $a \le b \le c < d$.

Case 3. $5 \le a \le b \le c = d$. Let $E_1 : v_1, v_2, \ldots, v_{a+2}$ be a path of order $a+2$. Let $E_2 : w_1, w_2, \ldots, w_{b-a+3}$ be a path of order $b - a + 3$. Let $E_i : z_i(3 \le i \le b-a+2)$ be a $b - a$ copies of K_1. Let $F_1 : s_1, s_2, \ldots, s_{c-b+1}$ be a path P_{c-b+1} of order $c-b+1$. Choose a vertex $x = v_i(1 \le i \le a+1)$ such that $d^*(v_{a+2}, x) = b-2$. We construct the graph G as follows: (i) identifying the vertices v_{a-1} and v_{a+1} of E_1 with w_1 and w_{b-a+3} of E_2 respectively and joining each E_i $(3 \le i \le b-a+2)$ with v_{a-1} and w_i. (ii) identify the vertices s_1 in F_1 and v_1 in E_1 and join each vertex $s_i(1 \le i \le c-b+2)$ (not each i, necessarily i is even) in F_1 with the vertex x. It is easy to verify that that $e_1(v_{a+2}) = a$, $e_1{}^*(v_{a+2}) = b$, $e_{\Delta f_1}(v_{a+2}) = c$, $e_{D_1}(v_{a+2}) = d$. It is easy to verify that there is no vertex x in G with $e_1(x) < a$, $e_1{}^*(x) < b$, $e_{\Delta f_1}(x) < c$ and $e_{D_1}(x) < d$. Also it is easy to verify that there is no vertex x in G with $e_1(x) > a$, $e_1{}^*(x) > b$, $e_{\Delta f_1}(x) > c$ and $e_{D_1}(x) > d$. Thus $diam_1(G) = a$, $diam_1{}^*(G) = b$, $diam_{\Delta f_1}(G) = c$ and $diam_{D_1}(G) = d$ as $a \le b \le c = d$.

References:

1. G. Chartrand and H. Escuadro, and P. Zhang, *Detour Distance in Graphs,* J.Combin. Math. Combin. Comput., 53(2005), 75–94.
2. G. Chartrand and G. L. Johns, and S. Tian, *Detour Distance in Graphs,* Annals of Discrete Mathematics, 55(1993), 127–136.
3. G. Chartrand and P. Zhang, Introduction to Graph Theory, Tata McGraw-Hill, New Delhi, 2006.
4. I.Keerthi Asir and S. Athisayanathan, *Triangle Free Detour Distance in Graphs,* J. Combin. Math. Combin. Comput., (Accepted).
5. I.Keerthi Asir and S. Athisayanathan, *Vertex-to-Clique Detour Distance in Graphs,* Journal of Prime Research In Mathematics, 12 (2016), 45–59.
6. I.Keerthi Asir and S. Athisayanathan, *Vertex-to-Clique Monophonic Distance in Graphs,* Ars Combinatoria, (Accepted).
7. A.P. Santhakumaran and S. Arumugam, *Centrality with respect to Cliques,* International Journal of Management and Systems, 18 (2002), 275-280.
8. A.P. Santhakumaran and P. Titus, *Monophonic Distance in Graphs,* Discrete Math. Algorithms Appl., 3(2011), 159–169.

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