ANALYSIS OF BATCH ARRIVAL BULK SERVICE QUEUE WITH THREE STAGES OF HETEROGENEOUS SERVICE MULTIPLE VACATION SETUP TIME AND BALKING

G. Ayyappan
Department of Mathematics, Pondicherry Engineering College,
Pillaichavady, Puducherry - 605 014, India
ayyappan@pec.edu
T. Deepa
Department of Mathematics, Pondicherry Engineering College,
Pillaichavady, Puducherry - 605 014, India
deepatmaths@gmail.com

Abstract: This paper studies an $M^{[x]}/G(a,b)/1$ queueing model with three stages of heterogeneous service multiple vacation setup time and balking. The service is done in bulk with a minimum of `a' customers and a maximum of `b' customers. At first the server provides first stage of bulk service to the arriving customers and sequentially, the server provides second and third stages of services. After completing third stage of service, if the queue size is less than `$a$', then the server leaves for a vacation of random length. When he returns from the vacation, if the queue length is still less than `a', he leaves for another vacation and so on. This process continues until he finds at least `$a$' customer in the queue. After the vacation completion moment, if the server finds at least `a' customer waiting for service, he requires to setup the system to start the service. After this setup he serves a batch of ξ customers $(a \le \xi \le b)$. Using supplementary variable technique, the probability generating function of the queue size, expected queue length, expected waiting time, expected busy period and expected idle period are derived. The analysis of cost is carried over. Numerical illustrations are also presented to visualize the effect of system parameters.

Keywords: Batch Arrival, Bulk Service, Three Stages Of Heterogeneous Service, Multiple Vacation, Setup Time, Balking.

AMS Subject Classification (2010): 60K25, 90B22, 68M20.

1. Introduction: Queueing models with batch arrival and bulk service have significant applications in various practical situations. Due to development of manufacturing industries and service sectors, the necessity of analyzing the bulk queueing models becomes unavoidable. The major applications of vacation bulk queueing models are in computer and communication systems, manufacturing systems, service systems, etc. In the vacation queueing model, the server is utilized for some other secondary jobs whenever the system becomes empty. Queueing models with vacations have been studied by many researchers for the past few decades. The arriving batch may join the queue for service or may not join the queue (i.e. balking) due to impatience.

A literature survey on vacation queueing models can be found in Doshi [7] and Takagi [19] which include some applications. Krishna Reddy et al. [12] considered an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacations, setup times and N policy. They derived the steady-state system size distribution, cost model, expected length of idle and busy period. Madan [13] derived the PGF of queue length for a single server non-Markovian queueing model where the server provides two stages of heterogeneous service. Arumuganathan and Jeyakumar [1] obtained the probability generating function of queue length distribution at an arbitrary time epoch and a cost model for the $M^{[X]}/G(a,b)/1$ queueing model. Madan and Choudhury [14] analyzed a single server queueing model with two phase of heterogeneous service and vacation. They obtained the steady state PGF of queue size at an arbitrary epoch. Arumuganathan and Judeth Malliga [2] carried over a steady-state analysis of a bulk queueing model with repair of service station and setup time. Also they derived various performance measures and performed cost analysis.

Choudhury et al. [4] carried over the steady state analysis of an $M^x/G/1$ queueing model where the server provides two phases of service and undergoes multiple vacation. Senthil Kumar and Jeyakumar[18] derived the PGF of orbit size, average size of the orbit and expected waiting time in the orbit for a single server retrial queueing model where the server provides the first essential service (FES) as well as the second essential service (SES) and then resumes vacation after completing the SES for a batch. Choudhury [5] analyzed an $M/G/1$ retrial queueing model with two phase of service and Bernoulli vacation. They derived the PGF of queue size at an arbitrary epoch for the steady state case. Moreno [15] presented a steady-state analysis of an Geo/G/1 queueing model with multiple vacation and setup-closedown times where he has derived a joint generating function of the server state and the system length using supplementary variable technique. He also studied the expected length of busy periods, expected waiting time in the queue and the system.

The $M^{[X]}/G(a,b)/1$ queueing model with closedown time and multiple vacation was investigated by Jeyakumar and Senthilnathan [11] where the server would be sent for repair only after the service completion for the batch in service, even if breakdown occurred. They also computed PGF of queue size and system performance measures like expected length of busy period and idle period, expected queue length and waiting time. Jeyakumar and Senthilnathan [10] derived the PGF of queue size for the $M^{[X]}/G(a,b)/1$ queueing system with setup time, closedown time and multiple vacation where the batch of customers in service would not be getting affected if breakdown occurs. Ramaswami and Jeyakumar [17] obtained the queue size distribution for the $M^{[X]}/G(a,b)/1$ queue with multiple vacation and state dependent arrivals using supplementary variable method.

Using supplementary variable technique, Rajadurai et al. [16] obtained the PGF of system size for a batch arrival retrial, feedback queue with two phases of service, Bernoulli vacation, negative customers and server breakdown. Using supplementary variable technique, Jeyakumar and Senthilnathan [9] obtained the PGF of queue size at an arbitrary epoch and at an completion epoch for a single server non-Markovian queueing model. Ayyappan and Shyamala [3] derived the PGF of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time for both steady state and transient cases. An overview on vacation queueing systems was presented by Upadhyaya [20]. Jeyakumar and Senthilnathan [8] investigated a single server batch arrival queueing model in which the server undergoes breakdown as well as multiple vacation. They computed the PGF of queue length at an arbitrary epoch.

The rest of the paper is organized as follows. In section 2, batch arrival bulk service queueing model with three stages of heterogeneous service, multiple vacation, setup time and balking is described and the steady-state system size equations are considered. In section 3, using supplementary variable technique, the probability generating function of the queue size, its steady state condition are derived and some particular cases are provided. In section 4, performance measures like expected length of busy period, expected length of idle period, expected queue length and expected waiting time are obtained. In section 5, the cost model is presented to provide the total average cost of the model. In section 6, the performance measures and total average cost are verified numerically with various system parameters. In section 7, this research work is concluded with the proposed future work.

2. Model Description and System Equations: An $M^{[x]}/G(a,b)/1$ queueing model with three stages of heterogeneous service, multiple vacation, setup time and balking is considered. The server provides first stage of bulk service to the arriving customers and then consecutively the server provides second and third stages of services. After completing third stage of service, if the queue size is less than `$a$', then the server leaves for a vacation of random length. When he returns from the vacation, if the queue length is still less than `a', he leaves for another vacation and so on. This process continues until he finds at least `$a$' customer in the queue. At the vacation completion moment, if the server finds at least `a' customer waiting for service, he requires to setup the system to start the service. After this setup, he serves a batch of ξ customers ($a \le \xi \le b$). The batch of arriving customers may join the queue with probability α or balk the queue with probability $1-\alpha$.

2.1 Notations: The following notations are used in this paper.

λ - Arrival rate,
X - Group size random variable,
$g_k - Pr\{X=k\}$,
$X(z)$ - Probability generating function (PGF) of X.

Here $S_1(.), S_2(.), S_3(.), V(.)$ and $G(.)$ represent the cumulative distribution function (CDF) of first service time, second service time, third service time, vacation time, setup time and their corresponding probability density functions are $s_1(x), s_2(x), s_3(x), v(x)$ and $g(x)$ respectively. $S_1^0(t), S_2^0(t), S_3^0(t), V^0(t)$ and $G(t)$ represent the remaining first service time of a batch, second service time of a batch, third service time of a batch, vacation time, and setup time at time t respectively. $\tilde{S}_1(\theta), \tilde{S}_2(\theta), \tilde{S}_3(\theta), \tilde{V}(\theta)$ and $\tilde{G}(\theta)$ represent the Laplace-Stieltjes transform of S_1, S_2, S_3, V and G respectively.

The supplementary variables $S_1^0(t), S_2^0(t), S_3^0(t), V^0(t)$ and $G^0(t)$ are introduced in order to obtain the bivariate Markov process $\{N(t), Y(t)\}$, where $N(t) = \{N_q(t) \cup N_s(t)\}$ and

$Y(t) = (0)[1](2)\{3\}\langle 4\rangle, \text{iftheserverison(firstservice)[secondservice](thirdservice)} \{\text{vacation}\}\langle\text{setuptime}\rangle,$
$Z(t) = j, \text{iftheserverisonj}^{\text{th}}\text{vacation},$
$N_s(t) = \text{Numberofcustomersintheserviceattimet},$
$N_q(t) = \text{Numberofcustomersinthequeueattimet}.$

Define the probabilities as,

$$P_{i,j}^{(1)}(x,t)dt = P\{N_s(t)=i, N_q(t)=j, x \le S_1^0(t) \le x+dx, Y(t)=0\}, a \le i \le b, j \ge 0,$$
$$P_{i,j}^{(2)}(x,t)dt = P\{N_s(t)=i, N_q(t)=j, x \le S_2^0(t) \le x+dx, Y(t)=1\}, a \le i \le b, j \ge 0,$$
$$P_{i,j}^{(3)}(x,t)dt = P\{N_s(t)=i, N_q(t)=j, x \le S_3^0(t) \le x+dx, Y(t)=2\}, a \le i \le b, j \ge 0,$$
$$Q_{j,n}(x,t)dt = P\{N_q(t)=n, x \le V^0(t) \le x+dx, Y(t)=3, Z(t)=j\}, n \ge 0, j \ge 1,$$
$$G_n(x,t)dt = P\{N_q(t)=n, x \le G^0(t) \le x+dx, Y(t)=4\}, n \ge 0.$$

The supplementary variable technique was introduced by Cox [6]. Using supplementary variables one can convert non-Markovian models into Markovian models. The steady-state system size equations are obtained as follows:

$$-P'^{(1)}_{i,0}(x) = -\lambda P_{i,0}^{(1)}(x) + \lambda(1-\alpha)P_{i,0}^{(1)}(x)$$
$$+G_i(0)s_1(x) + \sum_{m=a}^{b} P_{m,i}^{(3)}(0)s_1(x), \quad a \le i \le b, \tag{1}$$
$$-P'^{(1)}_{i,j}(x) = -\lambda P_{i,j}^{(1)}(x) + \lambda(1-\alpha)P_{i,j}^{(1)}(x) + \alpha\sum_{k=1}^{j} P_{i,j-k}^{(1)}(x)\lambda g_k, \quad a \le i \le b-1,\ j \ge 1, \tag{2}$$
$$-P'^{(1)}_{b,j}(x) = -\lambda P_{b,j}^{(1)}(x) + \lambda(1-\alpha)P_{b,j}^{(1)}(x) + G_{b+j}(0)s_1(x)$$
$$+\alpha\sum_{k=1}^{j} P_{b,j-k}^{(1)}(x)\lambda g_k + \sum_{m=a}^{b} P_{m,b+j}^{(3)}(0)s_1(x), \quad j \ge 1,` \tag{3}$$
$$-P'^{(2)}_{i,0}(x) = -\lambda P_{i,0}^{(2)}(x) + \lambda(1-\alpha)P_{i,0}^{(2)}(x) + P_{i,0}^{(1)}(0)s_2(x), \quad a \le i \le b, \tag{4}$$
$$-P'^{(2)}_{i,j}(x) = -\lambda P_{i,0}^{(2)}(x) + \lambda(1-\alpha)P_{i,j}^{(2)}(x) + P_{i,j}^{(1)}(0)s_2(x) + \alpha\sum_{j=1}^{k} P_{i,j-k}^{(2)}(x)\lambda g_k,$$
$$a \le i \le b,\ j \ge 1, \tag{5}$$
$$-P'^{(3)}_{i,0}(x) = -\lambda P_{i,0}^{(3)}(x) + \lambda(1-\alpha)P_{i,0}^{(3)}(x) + P_{i,0}^{(2)}(0)s_3(x), \quad a \le i \le b, \tag{6}$$
$$-P'^{(3)}_{i,j}(x) = -\lambda P_{i,0}^{(3)}(x) + \lambda(1-\alpha)P_{i,j}^{(2)}(x) + P_{i,j}^{(2)}(0)s_3(x) + \alpha\sum_{j=1}^{k} P_{i,j-k}(x)\lambda g_k,$$
$$a \le i \le b,\ j \ge 1, \tag{7}$$

$$-Q'_{1,0}(x) = -\lambda Q_{1,0}(x) + \lambda(1-\alpha)Q_{1,0}(x) + \sum_{m=a}^{b} P_{m,0}^{(3)}(0)v(x), \quad (8)$$

$$-Q'_{1,n}(x) = -\lambda Q_{1,n}(x) + \lambda(1-\alpha)Q_{1,n}(x) + \sum_{m=a}^{b} P_{m,n}^{(3)}(0)v(x)$$
$$+\alpha \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda g_k, \quad 1 \le n \le a-1, \quad (9)$$

$$-Q'_{1,n}(x) = -\lambda Q_{1,n}(x) + \lambda(1-\alpha)Q_{1,n}(x) + \alpha \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda g_k, \quad n \ge a, \quad (10)$$

$$-Q'_{j,0}(x) = -\lambda Q_{j,0}(x) + \lambda(1-\alpha)Q_{j,0}(x) + Q_{j-1,0}(0)v(x), \quad j \ge 2, \quad (11)$$

$$-Q'_{j,n}(x) = -\lambda Q_{j,n}(x) + \lambda(1-\alpha)Q_{1,n}(x) + Q_{j-1,n}(0)v(x) + \alpha \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda g_k, \quad j \ge 2,\ 1 \le n < a, \quad (12)$$

$$-Q'_{j,n}(x) = -\lambda Q_{j,n}(x) + \lambda(1-\alpha)Q_{j,n}(x) + \alpha \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda g_k,\ j \ge 2,\ n \ge a, \quad (13)$$

$$-G'_n(x) = -\lambda G_n(x) + \lambda(1-\alpha)G_n(x) + \alpha \sum_{k=1}^{n} G_{n-k}(x)\lambda g_k + \sum_{l=1}^{\infty} Q_{l,n}(0)g(x), \quad j \ge 2,\ n \ge a. \quad (14)$$

3. Probability Generating Function of Queue Size: The Laplace-Stieltjes transform of $\boldsymbol{P_{i,j}^{(1)}(x), P_{i,j}^{(2)}(x), Q_{j,n}(x), G_n(x)}$ are defined as follows:

$$\tilde{P}_{i,j}^{(1)}(\theta) = \int_0^{\infty} e^{-\theta x} P_{i,j}^{(1)}(x)dx,$$

$$\tilde{P}_{i,j}^{(2)}(\theta) = \int_0^{\infty} e^{-\theta x} P_{i,j}^{(2)}(x)dx,$$

$$\tilde{P}_{i,j}^{(3)}(\theta) = \int_0^{\infty} e^{-\theta x} P_{i,j}^{(3)}(x)dx,$$

$$\tilde{Q}_{j,n}(\theta) = \int_0^{\infty} e^{-\theta x} Q_{j,n}(x)dx, \quad \tilde{G}_n(\theta) = \int_0^{\infty} e^{-\theta x} G_n(x)dx.$$

Taking Laplace-Stieltjes transform from (1) to (14), we get

$$\theta\tilde{P}_{i,0}^{(1)}(\theta) - P_{i,0}^{(1)}(0) = \lambda\tilde{P}_{i,0}^{(1)}(\theta) - \lambda(1-\alpha)\tilde{P}_{i,0}^{(1)}(\theta) - \tilde{S}_1(\theta)\sum_{m=a}^{b} P_{m,i}^{(3)}(0) - \tilde{S}_1(\theta)G_i(0), \quad a \le i \le b, \quad (15)$$

$$\theta\tilde{P}_{i,j}^{(1)}(\theta) - P_{i,j}^{(1)}(0) = \lambda\tilde{P}_{i,j}^{(1)}(\theta) - \lambda(1-\alpha)\tilde{P}_{i,j}^{(1)}(\theta) - \alpha\sum_{k=1}^{j} \tilde{P}_{i,j-k}^{(1)}(\theta)\lambda g_k, \quad (16)$$

$$\theta\tilde{P}_{b,j}^{(1)}(\theta) - P_{b,j}^{(1)}(0) = \lambda\tilde{P}_{b,j}^{(1)}(\theta) - \lambda(1-\alpha)\tilde{P}_{b,j}^{(1)}(\theta) - \sum_{k=1}^{j} \tilde{P}_{b,j-k}(\theta)\lambda g_k$$
$$-\tilde{S}_1(\theta)G_{b+j}(0) - \tilde{S}_1(\theta)\sum_{m=a}^{b} P_{m,b+j}^{(3)}(0),\ j \ge 1, \quad (17)$$

$$\theta\tilde{P}_{i,0}^{(2)}(\theta) - P_{i,0}^{(2)}(0) = \lambda\tilde{P}_{i,0}^{(2)}(\theta) - \lambda(1-\alpha)\tilde{P}_{i,0}^{(2)}(\theta) - \tilde{S}_2(\theta)P_{i,0}^{(1)}(0), \quad (18)$$

$$\theta\tilde{P}_{i,j}^{(2)}(\theta) - P_{i,j}^{(2)}(0) = \lambda\tilde{P}_{i,j}^{(2)}(\theta) - \lambda(1-\alpha)\tilde{P}_{i,j}^{(2)}(\theta) - \tilde{S}_2(\theta)P_{i,0}^{(1)}(0) - \alpha\sum_{j=1}^{k} \tilde{P}_{i,j-k}^{(2)}(\theta)\lambda g_k, \quad (19)$$

$$\theta\tilde{P}_{i,0}^{(3)}(\theta) - P_{i,0}^{(3)}(0) = \lambda\tilde{P}_{i,0}^{(3)}(\theta) - \lambda(1-\alpha)\tilde{P}_{i,0}^{(3)}(\theta) - \tilde{S}_3(\theta)P_{i,0}^{(2)}(0), \quad (20)$$

$$\theta\tilde{P}_{i,j}^{(3)}(\theta) - P_{i,j}^{(3)}(0) = \lambda\tilde{P}_{i,j}^{(3)}(\theta) - \lambda(1-\alpha)\tilde{P}_{i,j}^{(3)}(\theta) - \tilde{S}_3(\theta)P_{i,0}^{(2)}(0) - \alpha\sum_{j=1}^{k} \tilde{P}_{i,j-k}^{(3)}(\theta)\lambda g_k, \quad (21)$$

$$\theta\tilde{Q}_{1,0}(\theta) - Q_{1,0}(0) = \lambda\tilde{Q}_{1,0}(\theta) - \lambda(1-\alpha)\tilde{Q}_{1,0}(\theta) - \sum_{m=a}^{b} P_{m,0}^{(3)}(0)\tilde{V}(\theta), \quad (22)$$

$$\theta\tilde{Q}_{1,n}(\theta) - Q_{1,n}(0) = \lambda\tilde{Q}_{1,n}(\theta) - \lambda(1-\alpha)\tilde{Q}_{1,n}(\theta) - \alpha\sum_{k=1}^{n} \tilde{Q}_{1,n-k}(\theta)\lambda g_k - \sum_{m=a}^{b} P_{m,n}^{(3)}(0)\tilde{V}(\theta), \quad 1 \le n \le a-1, \quad (23)$$

$$\theta\tilde{Q}_{1,n}(\theta) - Q_{1,n}(0) = \lambda\tilde{Q}_{1,n}(\theta) - \lambda(1-\alpha)\tilde{Q}_{1,n}(\theta) - \alpha\sum_{k=1}^{n} \tilde{Q}_{1,n-k}(\theta)\lambda g_k,\ n \ge a, \quad (24)$$

$$\theta\tilde{Q}_{j,0}(\theta) - Q_{j,0}(0) = \lambda\tilde{Q}_{j,0}(\theta) - \lambda(1-\alpha)\tilde{Q}_{j,0}(\theta) - \tilde{V}(\theta)Q_{j-1,0}(0), \quad (25)$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda\tilde{Q}_{j,n}(\theta) - \lambda(1-\alpha)\tilde{Q}_{j,n}(\theta) - \tilde{V}(\theta)Q_{j-1,n}(0) - \alpha\sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda g_k, \quad j \ge 2,\ 1 \le n \le a-1, \quad (26)$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda\tilde{Q}_{j,n}(\theta) - \lambda(1-\alpha)\tilde{Q}_{j,n}(\theta) - \alpha\sum_{k=1}^{n} \tilde{Q}_{j,n-k}(\theta)\lambda g_k,\ j \ge 2,\ n \ge a, \quad (27)$$

$$\theta\tilde{G}_n(\theta) - G_n(0) = \lambda\tilde{G}_n(\theta) - \lambda(1-\alpha)\tilde{G}_n(\theta) - \alpha\sum_{k=1}^{n} \tilde{G}_{n-k}(\theta)\lambda g_k - \sum_{l=1}^{\infty} \tilde{Q}_{l,n}(0)\tilde{G}(\theta),\ n \ge a. \quad (28)$$

To find the probability generating function (PGF), we define the following PGFs:

$$\tilde{P}_i^{(1)}(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}^{(1)}(\theta)z^j, \quad P_i^{(1)}(z,0) = \sum_{j=0}^{\infty} P_{ij}^{(1)}(0)z^j, \quad a \le i \le b,$$

$$\tilde{P}_i^{(2)}(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}^{(2)}(\theta)z^j, \quad P_i^{(2)}(z,0) = \sum_{j=0}^{\infty} P_{ij}^{(2)}(0)z^j, \quad a \le i \le b,$$

$$\tilde{P}_i^{(3)}(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}^{(3)}(\theta)z^j, \quad P_i^{(3)}(z,0) = \sum_{j=0}^{\infty} P_{ij}^{(3)}(0)z^j, \quad a \le i \le b,$$

$$\tilde{Q}_l(z,\theta)=\sum_{j=0}^{\infty}\tilde{Q}_{lj}(\theta)z^j,\quad Q_l(z,0)=\sum_{j=0}^{\infty}Q_{lj}(0)z^j,$$

$\tilde{G}(z,\theta)=\sum_{n=a}^{\infty}\tilde{G}_n(\theta)z^n,\quad G(z,0)=\sum_{n=a}^{\infty}G_n(0)z^n.$ (29)

By multiplying the equations from (15) to (28) by suitable power of z^n and summing over n, ($n=0$ to ∞) and using (29), we get

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{Q}_1(z,\theta)=Q_1(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(3)}(0)z^n,$ (30)

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{Q}_j(z,\theta)=Q_j(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\geq 2,$ (31)

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{G}(z,\theta)=G(z,0)-\tilde{G}(\theta)\sum_{n=a}^{\infty}\sum_{l=1}^{\infty}Q_{l,n}(0)z^n,$ (32)

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{P}_i^{(1)}(z,\theta)=P_i^{(1)}(z,0)-\tilde{S}_1(\theta)[\sum_{m=a}^{b}P_{m,i}^{(3)}(0)+G_i(0)],\ a\leq i\leq b-1,$ (33)

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{P}_b^{(1)}(z,\theta)=P_b^{(1)}(z,0)-\frac{\tilde{S}_1(\theta)}{z^b}[\sum_{m=a}^{b}P_m^{(3)}(z,0)-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}^{(3)}(0)z^j+G(z,0)-\sum_{j=a}^{b-1}G_j(0)z^j],$ (34)

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{P}_i^{(2)}(z,\theta)=P_i^{(2)}(z,0)-\tilde{S}_2(\theta)P_i^{(1)}(z,0),\ a\leq i\leq b,$ (35)

$(\theta-\lambda\alpha+\lambda\alpha X(z))\tilde{P}_i^{(3)}(z,\theta)=P_i^{(3)}(z,0)-\tilde{S}_3(\theta)P_i^{(2)}(z,0),\ a\leq i\leq b.$ (36)

Substitute $\theta=\lambda\alpha-\lambda\alpha X(z)$ in the equations from (30) to (36), we get

$Q_1(z,0)=\tilde{V}(\lambda\alpha-\lambda\alpha X(z))\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(3)}(0)z^n,$ (37)

$Q_j(z,0)=\tilde{V}(\lambda\alpha-\lambda\alpha X(z))\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\geq 2,$ (38)

$G(z,0)=\tilde{G}(\lambda\alpha-\lambda\alpha X(z))\sum_{n=a}^{\infty}\sum_{l=1}^{\infty}Q_{l,n}(0)z^n,$ (39)

$P_i^{(1)}(z,0)=\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))\left[\sum_{m=a}^{b}P_{m,i}^{(3)}(0)+G_i(0)\right],\ a\leq i\leq b-1,$ (40)

$P_i^{(2)}(z,0)=\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))P_i^{(1)}(z,0),$ (41)

$P_i^{(3)}(z,0)=\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))P_i^{(2)}(z,0),$ (42)

$P_b^{(1)}(z,0)=\frac{\left[\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))[\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))\times\sum_{m=a}^{b}P_m^{(1)}(z,0)+G(z,0)-\sum_{j=0}^{b-1}P_{m,j}^{(3)}(0)z^j-\sum_{j=0}^{b-1}G_j(0)z^j]\right]}{z^b-\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))}.$ (43)

Substitute equations from (37) to (43) in the equations from (30) to (36), we get

$\tilde{Q}_1(z,\theta)=\frac{[\tilde{V}(\lambda\alpha-\lambda\alpha X(z))-\tilde{V}(\theta)]\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}^{(3)}(0)z^n}{(\theta-\lambda\alpha-\lambda\alpha X(z))},$ (44)

$\tilde{Q}_j(z,\theta)=\frac{[\tilde{V}(\lambda\alpha-\lambda\alpha X(z))-\tilde{V}(\theta)]\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n}{(\theta-\lambda\alpha-\lambda\alpha X(z))},$ (45)

$\tilde{G}(z,\theta)=\frac{[\tilde{G}(\lambda\alpha-\lambda\alpha X(z))-\tilde{G}(\theta)][\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{n=0}^{a-1}\sum_{l=1}^{\infty}Q_{l,n}(0)z^n]}{(\theta-\lambda\alpha-\lambda\alpha X(z))},$ (46)

$\tilde{P}_i^{(1)}(z,\theta)=\frac{[\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))-\tilde{S}_1(\theta)][\sum_{m=a}^{b}P_{m,i}^{(3)}(0)+G_i(0)]}{(\theta-\lambda\alpha-\lambda\alpha X(z))},\ a\leq i\leq b-1,$ (47)

$\tilde{P}_b^{(1)}(z,\theta)=\frac{[\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))-\tilde{S}_1(\theta)]f(z)}{z^b-\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))},$ (48)

where

$$f(z)=\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))\sum_{m=a}^{b-1}P_m^{(1)}(z,0)+G(z,0)$$
$$-\sum_{j=0}^{b-1}P_{m,j}^{(3)}(0)z^j-\sum_{j=a}^{b-1}G_j(0)z^j.$$

$\tilde{P}_i^{(2)}(z,\theta)=\frac{[\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))-\tilde{S}_2(\theta)]P_i^{(1)}(z,0)}{(\theta-\lambda\alpha-\lambda\alpha X(z))},\ a\leq i\leq b,$ (49)

$\tilde{P}_i^{(3)}(z,\theta)=\frac{[\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))-\tilde{S}_3(\theta)]P_i^{(2)}(z,0)}{(\theta-\lambda\alpha-\lambda\alpha X(z))}.$ (50)

Let $\sum_{l=1}^{\infty}Q_{l,i}(0)=q_i,\ \sum_{m=a}^{b}P_{m,i}^{(3)}(0)=p_i,\ G_i(0)=g_i.$ (51)

3.1. PGF of Queue Size At An Arbitrary Time Epoch: If P(z) be the PGF of the queue size at an arbitrary time epoch, then

$P(z)=\sum_{m=a}^{b-1}\tilde{P}_m^{(1)}(z,0)+\tilde{P}_b^{(1)}(z,0)+\sum_{m=a}^{b}\tilde{P}_m^{(2)}(z,0)+\sum_{m=a}^{b}\tilde{P}_m^{(3)}(z,0)+\sum_{l=1}^{\infty}\tilde{Q}_l(z,0)+\tilde{G}(z,0).$ (52)

By substituting $\theta=0$ in the equations from (44) to (50), the equation (52) becomes

$$P(z) = \frac{[[\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))-1]\sum_{n=a}^{b-1}(z^b-z^n)p_n+[\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))-1]]}{(-\lambda\alpha+\lambda\alpha X(z))[z^b-\tilde{S}_1(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_2(\lambda\alpha-\lambda\alpha X(z))\tilde{S}_3(\lambda\alpha-\lambda\alpha X(z))]}. \tag{53}$$

Equation (53) has $2b$ unknowns $p_0, p_1, \ldots, p_{b-1}, q_0, q_1, \ldots, q_{a-1}$ and $g_a, g_{a+1}, \ldots, g_{b-1}$, we develop the following theorem to express q_n and g_n in terms of p_n in such a way that the numerator has only b unknowns. Now equation (53) gives the PGF of the number of customers involving only b unknowns.
By Rouche's theorem of complex variables, it can be proved that $z^b - \tilde{S}_1(\lambda\alpha - \lambda\alpha X(z))\tilde{S}_2(\lambda\alpha - \lambda\alpha X(z))\tilde{S}_3(\lambda\alpha - \lambda\alpha X(z))$ has $b-1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at this point, which gives b equations in b unknowns. We can solve these equations by any suitable numerical technique.

3.2 Steady-State Condition: The probability generating function has to satisfy $P(1) = 1$. In order to satisfy this condition, apply L' Hospital's rule and equating the expression to 1, it is derived that $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration, where $\rho = \lambda\alpha E(X)[E(S_1) + E(S_2) + E(S_3)]$.

Theorem 1: *Let q_n can be expressed in terms of p_n as*

$$q_n = \sum_{i=0}^{n} L_i p_{n-i}, \quad n = 0,1,2,,a-1, \tag{54}$$

where

$$L_n = \frac{\zeta_n + \sum_{i=1}^{n}\zeta_i L_{n-i}}{1-\zeta_0}, \quad n = 1,2,3,,a-1, \tag{55}$$

$$L_0 = \frac{\zeta_0}{1-\zeta_0}. \tag{56}$$

Proof: From equations (37) and (38), we have

$$\sum_{n=0}^{\infty} q_n z^n = \tilde{V}(\lambda\alpha - \lambda\alpha X(z))\left[\sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{a-1} q_n z^n\right]$$

$$= \sum_{j=0}^{\infty} \zeta_j z^j \left[\sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{a-1} q_n z^n\right]$$

$$= \sum_{n=0}^{a-1}\left[\sum_{i=0}^{n} \zeta_{n-i}(p_i + q_i)\right]z^n + \sum_{n=a}^{\infty}\left[\sum_{i=0}^{a-1} \zeta_{n-i}(p_i + q_i)\right]z^n. \tag{57}$$

Equating the coefficients of z^n, $n = 0,1,2,,a-1$ on both sides of (57), we get (54), (55) and (56).

Theorem 2 *The expression for g_i is given by*

$$g_i = \sum_{u=0}^{a-1}\left[\sum_{n=a}^{i} \eta_{i-n}\left(\zeta_{n-u} + \sum_{j=0}^{a-1-u} L_j \zeta_{n-j-u}\right)\right]p_u, \tag{58}$$

where η_i is the probability that i customers arrive during a setup time.
Proof: Using equations (39), we get

$$G(z,0) = \tilde{G}(\lambda\alpha - \lambda\alpha X(z))\sum_{n=a}^{\infty} q_n z^n = \left[\sum_{n=a}^{\infty} \eta_n z^n\right]\left[\sum_{n=a}^{\infty} q_n z^n\right].$$

Equating the coefficient of z^i, $i = a, a+1, a+2,, b-1$, we get

$$g_i = \sum_{n=a}^{i} q_n \eta_{i-n}, \quad i = a, a+1, a+2,, b-1. \tag{59}$$

From (54) and (59), we can obtain (58).

3.3 Particular Cases:

Case (i): When the server provides single stage of service, there is no balking and setup time, the equation (53) reduces into

$$P(z) = \frac{[(\tilde{S}(\lambda-\lambda X(z))-1)\sum_{n=a}^{b-1}(z^b-z^n)p_n+(z^b-1)\tilde{V}(\lambda-\lambda X(z))\times\sum_{n=0}^{a-1}p_n z^n+(z^b-1)(\tilde{V}(\lambda-\lambda X(z))-1)\sum_{n=0}^{a-1}q_n z^n]}{(-\lambda+\lambda X(z))(z^b-\tilde{S}(\lambda-\lambda X(z)))},$$

which coincides with the PGF of Arumuganathan and Jeyakumar [1] if the closedown time is zero and $N = a$.
Case (ii): When there is single stage of service and no balking is considered, the equation (53) reduces into

$$P(z)=\frac{[(\tilde{S}(\lambda-\lambda X(z))-1)\sum_{n=a}^{b-1}(z^b-z^n)p_n+(\tilde{S}(\lambda-\lambda X(z))-1)\sum_{n=a}^{b-1}(z^b-z^n)g_n+(\tilde{V}(\lambda-\lambda X(z))\tilde{G}(\lambda-\lambda X(z))-1)(z^b-1)\sum_{n=0}^{a-1}p_n z^n]}{(-\lambda+\lambda X(z))(z^b-\tilde{S}(\lambda-\lambda X(z)))},$$

which coincides with the PGF of Jeyakumar and Senthilnathan [11] if the closedown time and renovation are zero.

4. Performance Measures: In this section, the expressions for various performance measures like expected length of busy period, expected length idle period, expected queue length, expected waiting time are derived.

4.1 Expected Length of Busy Period

Theorem 3 *Let B be the busy period random variable. Then the expected length of busy period is*

$$E(B)=\frac{E(S)}{\sum_{n=0}^{a-1}p_n}, \qquad (60)$$

where $E(S)=E(S_1)+E(S_2)+E(S_3)$.

Proof: Let S be the residence time that the server is rendering first stage of service or second stage of service or third stage of service.

$E(S)=E(S_1)+E(S_2)+E(S_3)$.

Define a random variable J_1 as

$$J_1=\begin{cases}0, & \text{if the server finds less than 'a' customers in the queue at the third stage} \\ & \text{completion epoch,} \\ 1, & \text{if the server finds atleast 'a' customers at the third stage completion epoch.}\end{cases}$$

Now the expected length of the busy period is given by

$$E(B)=E(B/J=0)P(J=0)+E(B/J=1)P(J=1)$$
$$=E(S)P(J=0)+[E(S)+E(B)]P(J=1),$$

where $E(S)$ is the mean service time.

Solving for $E(B)$, we get

$$E(B)=\frac{E(S)}{P(J_1=0)}=\frac{E(S)}{\sum_{n=0}^{a-1}p_n}.$$

4.2 Expected Length of Idle Period:

Theorem 4 *Let I be the idle period random variable. Then the expected length of idle period is given by*

$$E(I)=E(I_1)+E(G), \qquad (61)$$

$$E(I_1)=\frac{E(V)}{1-\sum_{n=0}^{a-1}\sum_{i=0}^{n}\zeta_{n-i}k_i},$$

where I_1 is the idle period due to multiple vacation process and $E(G)$ is the expected setup time.

Proof: Define a random variable J_2 as

$$J_2=\begin{cases}0, & \text{if the server finds atleast 'a' customers after first vacation,} \\ 1, & \text{if the server finds less than 'a' customers after first vacation.}\end{cases}$$

Now the expected length of idle period is given by

$$E(I)=E(I_1/J_2=0)P(J_2=0)+E(I_1/J_2=1)P(J_2=1)$$
$$=E(V)P(J_2=0)+[E(V)+E(I_1)]P(J_2=1).$$

Solving for $E(I_1)$, we have

$$E(I_1)=\frac{E(V)}{P(J_2=0)}.$$

From equation (37), we get

$$P(J_2=0)=1-P(J_2=1)$$
$$=1-\sum_{n=0}^{a-1}Q_{1n}(0)$$
$$=1-\sum_{n=0}^{a-1}\sum_{i=0}^{n}\zeta_{n-i}k_i.$$

4.3 Expected Queue Length: The expected queue length E(Q) at an arbitrary epoch is obtained by differentiating P(z) at z=1 and is given by

$$E(Q) = \frac{\left[f_1 \sum_{n=a}^{b-1}(b-n)(p_n+g_n)+f_2 \sum_{n=a}^{b-1}(b(b-1)-n(n-1))(p_n+g_n)+f_3 \sum_{n=0}^{a-1}(q_n+p_n)+f_4 \sum_{n=0}^{a-1} p_n+f_5 \sum_{n=0}^{a-1}(nq_n+np_n)+f_6 \sum_{n=0}^{a-1} np_n\right]}{2.((\lambda.\alpha.X_1).(b-S_1^{(1)}-S_2^{(1)}-S_3^{(1)}))^2}, \tag{62}$$

where $f_1 = H_4.H_1 - H_3.H_2$,

$f_2 = H_3.H_1$,

$f_3 = b.(b-1).V1.H1 + 2b.G1.V1.H1 + b.V2.H1 - b.V1.H2$,

$f_4 = b.(b-1).G1.H1 + b.G2.H1 - b.G1.H2$,

$f_5 = 2.b.V1.H1$,

$f_6 = 2.b.G1.H1$,

$H_1 = \lambda.\alpha.X_1.(b - S_1^{(1)} - S_2^{(1)} - S_3^{(1)})$,

$H_2 = (\lambda.\alpha.X_2).(b - S_1^{(1)} - S_2^{(1)} - S_3^{(1)})$

$+(\lambda.\alpha.X_1)(b(b-1) - S_1^{(2)} - S_2^{(2)} - S_3^{(2)} - 2.S_1^{(1)}.S_2^{(1)} - 2.S_1^{(1)}.S_3^{(1)} - 2.S_2^{(1)}.S_3^{(1)})$,

$H_3 = S_1^{(1)} + S_2^{(1)} + S_3^{(1)}$,

$H_4 = S_1^{(2)} + S_2^{(2)} + S_3^{(2)} + 2.S_1^{(1)}.S_2^{(1)} + 2.S_1^{(1)}.S_3^{(1)} + 2.S_2^{(1)}.S_3^{(1)}$.

and

$X_1 = E(X)$, $S_1^{(1)} = \lambda.\alpha.X1.E(S_1)$, $S_2^{(1)} = \lambda.\alpha.X1.E(S_2)$, $S_3^{(1)} = \lambda.\alpha.X1.E(S_3)$,

$S_1^{(2)} = \lambda.\alpha.X2.E(S_1) + \lambda^2.\alpha^2.(E(X))^2.E(S_1^2)$,

$S_2^{(2)} = \lambda.\alpha.X2.E(S_2) + \lambda^2.\alpha^2.(E(X))^2.E(S_2^2)$,

$S_3^{(2)} = \lambda.\alpha.X2.E(S_3) + \lambda^2.\alpha^2.(E(X))^2.E(S_3^2)$,

$V1 = \lambda.\alpha.X1.E(V)$, $V2 = \lambda.\alpha.X2.E(V) + \lambda^2.\alpha^2.(E(X))^2.E(V^2)$,

$G1 = \lambda.\alpha.X1.E(G)$, $G2 = \lambda.\alpha.X2.E(G) + \lambda^2.\alpha^2.(E(X))^2.E(G^2)$.

4.4 Expected Waiting Time: The expected waiting time is obtained by using Little's formula as: $\boldsymbol{E(W) = \frac{E(Q)}{\lambda E(X)}}$,
where $E(Q)$ is given in (62).

5. Cost Model: We obtain the total average cost with the following assumptions:

C_s - startup cost
C_h - holding cost per customer
C_o - operating cost per unit time
C_r - the reward per unit time due to vacation
C_g - setup cost per unit time

The length of the cycle is the sum of the idle period and busy period.
The expected length of cycle $E(T_c)$, is obtained as,

$$E(T_c) = E(I) + E(B) = \frac{E(V)}{P(U=0)} + E(G) + \frac{E(T)}{\sum_{n=0}^{a-1} p_n}.$$

The total average cost per unit is given by

Total average cost = Start − up cost percycle + Holding cost ofcustomers in the queue
+Operating cost. ρ
+setup time cost − reward due to vacation per cycle.

$$= \left[C_s + C_g E(G) - C_r.\frac{E(V)}{P(U=0)}\right]\frac{1}{E(T_c)}$$
$$+C_h.E(Q) + C_0.\rho,$$

where $\rho = \lambda\alpha E(X)[E(S_1) + E(S_2) + E(S_3)]/b$.

6. Numerical Illustration: In this section a numerical example is analyzed using MATLAB, the zeroes of the function $z^b - \tilde{S}_1(\lambda\alpha - \lambda\alpha X(z))\tilde{S}_2(\lambda\alpha - \lambda\alpha X(z))\tilde{S}_3(\lambda\alpha - \lambda\alpha X(z))$ are obtained and simultaneous equations are solved.

1. Batch size distribution of the arrival is Geometric with mean two.
2. Service time distribution is Erlang –k with $k = 2$.
3. Vacation time, setup time are exponential with parameter $\beta = 9$ and $\gamma = 7$ respectively.
4. $\alpha = 0.2$.
5. Start-up cost: Rs.5

Holding cost per customer: Rs. 0.75

Operating cost per unit time: Rs. 6
Reward per unit time due to vacation: Rs.3
Setup time cost per unit time: Rs. 0.50
Additional service cost per unit time: Rs.0.80

Table 1: Arrival rate vs Total average cost and performance measures
$\mu_1 = 6, \mu_2 = 7, \mu_3 = 8, a = 2, b = 4$'

λ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$	TAC
1.0	2.1015	1.0507	1.1512	0.2858	5.657
1.5	3.0170	1.0257	1.3241	0.2806	5.8168
2.0	3.9979	1.0416	1.7686	0.2708	7.1786
2.5	5.3093	1.0619	2.2080	0.2630	7.7936
3.0	7.0093	1.1682	3.1175	0.2569	8.5591
3.5	9.2513	1.3517	4.1030	0.2548	10.0457
4.0	12.5245	1.6656	5.1126	0.2513	13.6646
4.5	17.3864	2.1237	6.2251	0.2498	15.9856
5.0	25.6907	2.5691	7.4231	0.2465	16.7589

Table 2: Arrival rate vs Total Average Cost and Performance Measures
$\mu_1 = 9, \mu_2 = 10, \mu_3 = 11, a = 2, b = 4$

λ	$E(Q)$	$E(W)$	$E(B)$	$E(I)$	TAC
1.0	1.5049	0.7524	0.0157	0.2991	4.0125
1.5	2.0128	0.7758	0.4148	0.2932	5.2345
2.0	2.9693	0.7823	0.7729	0.2907	6.1949
2.5	3.5346	0.7997	1.0018	0.2803	6.9522
3.0	4.8264	0.9044	1.3538	0.2727	7.2620
3.5	7.3456	1.0002	1.9149	0.2670	9.1249
4.0	9.4778	1.2847	2.9446	0.2645	11.6844
4.5	13.1245	1.7483	4.2353	0.2612	12.7683
5.0	18.0269	2.4027	6.4776	0.2567	14.4924

In Table 1, for $\mu_1 = 5, \mu_2 = 6, \mu_3 = 7, a = 2, b = 4$, we obtained the values of expected queue length, expected waiting time, expected busy period, expected idle period and total average cost. From Table 1, it is clear that if arrival rate λ increases, the mean queue size, expected waiting time, expected busy period and total average cost increase whereas expected idle period decreases.
In Table 2, for $\mu_1 = 7, \mu_2 = 8, \mu_3 = 9, a = 2, b = 4$, we obtained the values of expected queue length, expected waiting time, expected busy period, expected idle period and total average cost. It is evident from Table 2 that the mean queue size, expected waiting time, expected busy period and total average cost increase and expected idle period decreases with the increase of arrival rate λ. From Table 1 and 2, it is clear that expected queue length, expected waiting time, expected busy period, expected idle period and total average cost are all decrease, if the service rates increases.

7. Conclusion and Future Work: In this paper, we have derived the PGF of the queue size for an $\boldsymbol{M^{[X]}/G(a,b)/1}$ queueing model with three stages of heterogeneous service, multiple vacation, setup time and balking for the steady-state case. We have obtained various performance measures and have verified them numerically. Further, a cost analysis is also presented for the proposed model. In future, this work may be extended into a queueing model with multi stages of service, modified vacation and closedown.

References:

1. Arumuganathan, R. and Jeyakumar, S. (2005) `Steady state analysis of a bulk queue with multiple vacations, setup times with N-policy and closedown times', *Applied Mathematical Modelling*, Vol. 29, pp.972-986.

2. Arumuganathan, R. and Judeth Malliga, T. (2006) `Analysis of a bulk queue with repair of service station and setup time', *International Journal of Canadian Applied Mathematics Quarterly*, Vol. 13, No. 1, pp.19-42.
3. Ayyappan, G. and Shyamala, S. (2016) `Transient solution of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time', *International Journal of Operational Research*, Vol. 25, No. 2, pp.196-211.
4. Choudhury, G., Tadj, L. and Paul, M. (2007) `Steady state analysis of an $M^{x}/G/1$ queue with two phases of service and Bernoulli vacation schedule under multiple vacation policy', *Applied Mathematical Modelling*, Vol. 31, pp.1079-1091.
5. Choudhury, G. (2008) `Steady state analysis of an M/G/1 queue with linear retrial policy and two phase service under Bernoulli vacation schedule', *Applied Mathematical Modelling*, Vol. 32, pp.2480-2489.
6. Cox, D. R. (1965) `The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables', *Proceedings of Computer Philosophical Society*, Vol. 51, pp.433-441.
7. Doshi, B. T. (1986) `Queueing systems with vacations: a survey', *Queueing systems*, Vol. 1, pp.29-66.
8. Jeyakumar, S. and Senthilnathan, B. (2017) `Modelling and analysis of bulk service queueing model with multiple working vacations and server breakdown', *RAIRO - Operations Research*, Vol. 51, No. 2, pp.485-508.
9. Jeyakumar, S. and Senthilnathan, B. (2016) `Steady state analysis of bulk arrival and bulk service queueing model with multiple working vacations', *International Journal of Mathematics in Operational Research*, Vol. 9, No. 3, pp.375-394.
10. Jeyakumar, S. and Senthilnathan, B. (2014) `Modelling and analysis of a $M^{X}/G(a,b)/1$ queue with multiple vacations, setup time, closedown time and server breakdown without interruption', *International Journal of Operational Research*, Vol. 19, No. 1, pp.114-139.
11. Jeyakumar, S. and Senthilnathan, B. (2012) `A study on the behavior of the server breakdown without interruption in a $M^{X}/G(a,b)/1$ queueing system with multiple vacations and closedown time', *Applied Mathematics and Computation*, Vol. 219, pp.2618â€“2633.
12. Krishna Reddy, G. V., Nadarajan, R. and Arumuganathan, R. (1998) `Analysis of a bulk queue with N policy multiple vacations and setup times', *Computers Operations Research*, Vol. 25, No. 11, pp.957-967.
13. Madan, K. C. (2001) `On a single server queue with two stage heterogeneous service and deterministic vacation', *International Journal of Systems Science*, Vol. 32, No. 7, pp.837-844.
14. Madan, K.C. and Choudhury, G. (2005) `A single server queue with two phases of heterogeneous service under Bernoulli schedule and a generalized vacation time', *International Journal of Information and Management Sciences*, Vol. 16, No. 2, pp. 1-16.
15. Moreno, P. (2009) `A discrete-time single-server queueing system under multiple vacations and setup-closedown times', *Stochastic Analysis and Applications*, Vol. 27, No. 2, pp.221-239.
16. Rajadurai, P., Chandrasekaran, V. M. and Saravanarajan, M. C. (2015) `Steady state analysis of batch arrival feedback retrial queue with two phase service, negative customers, Bernoulli vacation and random breakdown', *International Journal of Mathematics in Operational Research*, Vol. 7, No. 5, pp.519-546.
17. Ramaswami, K. S. and Jeyakumar, S. (2014) `Non-Markovian bulk queueing system with state dependent arrivals and multiple vacations - a simulation approach', *International Journal of Mathematics in Operational Research*, Vol. 6, No. 3, pp.344-356.
18. Senthil Kumar, M. and Arumuganathan, R. (2008) `On the single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service', *Quality Technology and Quantitative Management*, Vol. 5, No. 2, pp.145-160.
19. Takagi, H., Queueing analysis: a foundation of performance evaluation, Vacations and Priority Systems, Part-1, vol. I, North Holland, 1991.
20. Upadhyaya, S. (2016) `Queueing systems with vacation: an overview', *International Journal of Mathematics in Operational Research*, Vol. 9, No. 2, pp.167-213.