A PERISHABLE (s,S) INVENTORY SYSTEM WITH AN INFINITE ORBIT AND RETRIALS

P. S. Reshmi

Research scholar, P. G. & Research Dept. of Mathematics, St.Peter's College, Kolenchery-682 311, Kerala, India.

K. P. Jose

Associate Professor, P. G. & Research Dept. of Mathematics, St.Peter's College, Kolenchery-682 311, Kerala, India.

Abstract: In this article, we consider a continuous review perishable (s,S) inventory system with instantaneous service and retrials. We assume that the arrival of customers constitutes a Poisson process. The maximum storage of inventory is S units and life time of each item is exponentially distributed. If the primary demand finds inventory level dry, then it is directed to an orbit of infinite capacity. The retrial demand from orbit follows exponetial distribution with linear rate. Whenever the on-hand inventory level drops to a prefixed level s $(0 \leq s < S)$, an order for replenishment is placed. The lead time is exponentially distributed. We derived the stability condition and analysed the system using Matrix Analytic Method. Various system performance measures are obtained and a suitable cost function for minimum expected cost is also derived.

Keywords: Cost Analysis, Matrix Analytic Method, Perishable Inventory, Retrials.

Introduction: For the last three decades, there were only a few studies on perishable inventory system. A class of perishable items includes medicines, fruits, vegetables and many types of packaged foods. The review article Nahmias [7] contributes a remarkable summaries of particular modelling efforts in perishable inventory. Ravichandran [11] explored the cost expression for a continuous review (S,s) ordering policy inventory system of perishable items with Erlangian life, in the stationary case. The cost expression is closely related to the stationary distribution of the stochastic process representing the inventory at any time. Jeganathan and Periyasamy [1] considered a continuous review perishable inventory system with a service facility. They assumed that the service may interrupt due to some physical phenomena and the service resumes after repair. Kalpakam and Sapna [3] studied an (s,S) perishable system with Poisson demands and exponentially distributed lead-times for items with exponential lifetimes. Perry and Stadje [10] discussed an inventory system for perishable commodities with finite shelf size and finite waiting room for demands. Krishnamoorthy and Jose [5] studied an (s,S) inventory system with positive lead-time and retrial of customers. They constructed a suitable cost function and analyzed its convexity. Krishnamoorthy and Anbazhagan [4] analyzed a perishable stochastic inventory system under continuous review at a service facility in which the waiting hall for customers is of finite size. Sivakumar [12] considered a continuous review perishable inventory system with a finite number of homogeneous sources of demands. Kumar and Elango [6] considered a single server queueing inventory system with finite waiting space. They discussed the problem as a Markov decision model and obtained the minimal average cost of the service using value iteration algorithm. The unique equilibrium probability distribution is also obtained in Matrix geometric form in which the two dimensional state space contains infinite queue length and finite capacity of inventory. Kalpakam and Arivarignan [2] analysed a continuous review inventory system in which items are removed from stock one at a time either due to random demand or random failure of item.

The paper is organized as follows. Section 2 explains mathematical modeling and analysis of the system. Section 3 describes the steady state probability vector. Algorithmic analysis is included in section 4.

System performance measures are obtained in section 5. Section 6 contains the cost analysis of the system.

Mathematical Modeling and Analysis*:* The following are the assumptions and notations used in this model.

Assumptions:

1. Inter-arrival times of primary demands are exponentially distributed with parameter α.
2. Retrial demands are exponentially distributed with parameter $i\beta$, when there are i customers in the orbit.
3. Life time of each item is exponentially distributed with rate ω.
4. Lead time is exponentially distributed with parameter θ.

Notations:

$N(t)$: Number of customers in the orbit at time t.

$I(t)$: Inventory level at time t.

e: $(1,1,...,1)'$, column vector of $1'$s of size $(S+1)$.

Then $\{(N(t), I(t)); t \geq 0\}$ is a Level Dependent Quasi-Birth Process on the state space $\{(i, j); i \geq 0, 0 \leq j \leq S\}$. Now the infinitesimal generator of the process is

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where the blocks $B_0, A_0, A_{1,i}$ and $A_{2,i}\,(i \geq 0)$ are given by

$$B_0 = \begin{bmatrix} \chi_0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \theta \\ \eta_1 & \chi_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \theta \\ 0 & \eta_2 & \chi_2 & \dots & 0 & 0 & 0 & \dots & 0 & \theta \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \eta_s & \chi_s & 0 & \dots & 0 & \theta \\ 0 & 0 & 0 & \dots & 0 & \eta_{s+1} & -\eta_{s+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \eta_S & -\eta_S \end{bmatrix}_{(S+1)\times(S+1)}$$

where

$$\chi_n = -(\alpha + n\omega + \theta); n = 0,1,\ldots,s$$

$$\eta_n = (\alpha + n\omega); n = 1,2,\ldots,S$$

$$A_0 = \begin{bmatrix} \alpha & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(S+1)\times(S+1)}$$

For $i \geq 1$,

$$A_{1,i} = \begin{bmatrix} \sigma_0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \theta \\ \eta_1 & \sigma_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \theta \\ 0 & \eta_2 & \sigma_2 & \dots & 0 & 0 & 0 & \dots & 0 & \theta \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \eta_s & \sigma_s & 0 & \dots & 0 & \theta \\ 0 & 0 & 0 & \dots & 0 & \eta_{s+1} & \sigma_{s+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \eta_S & \sigma_S \end{bmatrix}_{(S+1)\times(S+1)}$$

where

$$\sigma_n = \begin{cases} \chi_0, & n = 0 \\ -(\alpha + n\omega + i\beta + \theta), & n = 1, 2, \dots, s \\ -(\alpha + n\omega + i\beta), & n = s+1, s+2, \dots, S \end{cases}$$

$$A_{2,i} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ i\beta & 0 & 0 & \dots & 0 & 0 \\ 0 & i\beta & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & i\beta & 0 \end{bmatrix}_{(S+1)\times(S+1)}$$

Neuts-Rao truncation modifies the generator Q to the following form, where $A_{1,i} = A_1$ and $A_{2,i} = A_2$ for $i \geq N$: For the detailed discussion of the truncation, one can refer [8]

$$Q = \begin{bmatrix} B_0 & A_0 & & & & & & & \\ A_{2,1} & A_{1,1} & A_0 & & & & & & \\ & A_{2,2} & A_{1,2} & A_0 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & A_{2,N-1} & A_{1,N-1} & A_0 & & & \\ & & & & A_2 & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (1)$$

Define $A = A_0 + A_1 + A_2$. Then

$$A=\begin{bmatrix} -\theta & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & \theta \\ \phi_1 & \psi_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & \theta \\ 0 & \phi_2 & \psi_2 & \dots & \dots & \dots & \dots & \dots & 0 & \theta \\ \vdots & \vdots & \ddots & \ddots & \dots & \dots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \phi_s & \psi_s & 0 & \dots & 0 & \theta \\ 0 & 0 & 0 & \dots & 0 & \phi_{s+1} & -\phi_{s+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \phi_S & -\phi_S \end{bmatrix}_{(S+1)\times(S+1)}$$

where

$\phi_n = \alpha + n\omega + N\beta; 1 \le \text{n} \le \text{S}$

$\psi_n = -(\alpha + n\omega + N\beta + \theta); 1 \le \text{n} \le \text{s}$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots, \pi_S)$ be the steady state probability vector satisfying $\boldsymbol{\pi}A = 0$ and $\boldsymbol{\pi}\mathbf{e} = 1$, then

$$\pi_n = \begin{cases} \dfrac{\theta}{\phi_n}\pi_0; & n = 1 \\ (-1)^{n-1}\dfrac{\theta}{\phi_n}\displaystyle\prod_{k=1}^{n-1}\frac{\psi_k}{\phi_k}\pi_0; & n = 2,3,\dots,s \\ (-1)^{s}\dfrac{\theta}{\phi_n}\displaystyle\prod_{k=1}^{s}\frac{\psi_k}{\phi_k}\pi_0; & n = s+1, s+2,\dots,S \end{cases}$$

$$\pi_0 = \left[(-1)^s \prod_{k=1}^{s}\left(\frac{\psi_k}{\phi_k}\right)\left(1 + \sum_{m=s+1}^{S}\frac{\theta}{\phi_m}\right)\right]^{-1}$$

The system is stable if $\rho(N) \quad < 1$, where

$$\rho(N) \quad = \left(1 + \frac{\alpha}{N\beta}\right)\left[(-1)^s \prod_{k=1}^{s}\left(\frac{\psi_k}{\phi_k}\right)\left(1 + \sum_{m=s+1}^{S}\frac{\theta}{\phi_m}\right)\right]^{-1}.$$

From the well known result (see Neuts [9]) on positive recurrence of Q which states that $\boldsymbol{\pi}A_0\mathbf{e} < \boldsymbol{\pi}A_2\mathbf{e}$, the result follows.

Steady State Probability Vector: ***Let*** $\mathbf{x} = (x_0, x_1, \dots, x_{N-1}, x_N, \dots)$ ***be the steady state probability vector of*** Q***. Under the stability condition,*** x_i's $(i \ge N)$ ***are given by***

$x_{N+r-1} = x_{N-1}R^r \, (r \ge 1)$

where R is the unique non-negative solution of the equation

$R^2A_2 + RA_1 + A_0 = 0$

for which the spectral radius is less than 1 and the vectors $x_0, x_1, \dots, x_{N-1}$ are obtained by solving

$$\left.\begin{aligned} & x_0B_0 + x_1A_{2,1} = 0 \\ & x_{i-1}A_0 + x_iA_{1,i} + x_{i+1}A_{2,i+1} = 0; (1 \le i \le N-2) \\ & x_{N-2}A_0 + x_{N-1}(A_{1,N-1} + RA_2) = 0 \end{aligned}\right\} \quad (2)$$

subject to the normalizing condition

$$\left[\sum_{i=1}^{N-2} x_i + x_{N-1}(I-R)^{-1}\right]\mathbf{e} = 1 \qquad (3)$$

Algorithmic Analysis: The rate Matrix R is given by
$R = \lim_{n\to\infty} R_n(N)$,where $R_{n+1}(N) = -\left(R_n^2(N)A_2(N) - A_0(N)\right)A_1^{-1}(N)$
and $R_0(N) = 0$, where N can be chosen such that $|\nu(N) - \nu(N+1)| < \varepsilon$, ε is an arbitrary constant and $\nu(N)$, the spectral radius of R.

Calculation of Boundary Probabilities: Let $\mathbf{x}^* = (x_0, x_1, \ldots, x_{N-1})$ be the probability vector corresponding to the boundary portion of Q as in (1). Then $\mathbf{x}^*$ is the stationary vector of the infinitesimal generator T given below

$$T = \begin{bmatrix} B_0 & A_0 & & & & \\ A_{2,1} & A_{1,1} & A_0 & & & \\ & A_{2,2} & A_{1,2} & A_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{2,N-2} & A_{1,N-2} & A_0 \\ & & & & A_{2,N-1} & A_{1,N-1}+RA_2 \end{bmatrix}$$

Now the system (2) can be written as $\mathbf{x}^* T = 0.$ To solve this, we use the block Gauss-Seidel iterative method. The vectors $x_0, x_1, \ldots, x_{N-1}$ in the $(n+1)$ th iteration are given by

$$x_0(n+1) = x_1(n)A_{2,1}B_0^{-1}$$
$$x_i(n+1) = [x_{i+1}(n)A_{2,i+1} + x_{i-1}(n+1)A_0]A_{1,i}^{-1};\quad (1 \le i \le N-2)$$
$$x_{N-1}(n+1) = -x_{N-2}(n+1)A_0(A_{1,N-1} + RA_2)^{-1}$$

Each iteration is subject to the normalizing condition (3).

System Performance Measures : We partition the components of $\mathbf{x}$ as $x_i = (y_{i,0}, y_{i,1}, \ldots, y_{i,S})(i \ge 0)$. Then important performance measures of the system under steady state are

1. Expected inventory level, E_I, is given by $E_I = \sum_{j=1}^{S}\sum_{i=0}^{\infty} y_{i,j}$
2. Expected perishable rate, E_{PR}, is given by $E_{PR} = \omega\sum_{j=1}^{S} j\sum_{i=0}^{\infty} y_{i,j}$
3. Expected number of customers in the orbit, E_{CO}, is given by $E_{CO} = \sum_{j=1}^{S}\sum_{i=0}^{\infty} i y_{i,j}$
4. Expected reorder level, E_{RO}, is given by $E_{RO} = \sum_{i=0}^{\infty}(s+1)\omega y_{i,s+1} + \sum_{i=0}^{\infty}\alpha y_{i,s+1} + \sum_{i=1}^{\infty} i\beta y_{i,s+1}$
5. The overall rate of retrial is given by $E_R = \theta\sum_{j=0}^{S}\sum_{i=0}^{\infty} y_{i,j}$
6. The successful rate of retrials is given by $E_{SR} = \theta\sum_{j=1}^{S}\sum_{i=0}^{\infty} y_{i,j}$

7. Fraction of successful rate of retrials is $F_{SR} = \frac{E_{SR}}{E_R}$

Cost Analysis : Define the expected total cost of the system per unit time as

$$ETC = c_h E_I + c_s E_{RO} + c_f E_{PR} + c_b E_{CO}$$

where,

c_h : The inventory holding cost/unit/unit time c_s : The setup cost/order

c_f : The failure cost/unit/unit time

c_b : backorder cost of a demand in the orbit

/unit time.

Now, one can find minimum value of expected total cost per unit time by varying the different parameters α, β, ω and θ.

Concluding Remarks: In this article, we studied a continuous review perishable (s, S) inventory system with infinite orbit and retrials. The primary demand constitutes a Poisson process. Assumptions made on retrial demand, life time, lead-time are exponential distributions. We analyzed the system using Matrix Analytic Method. Stability condition and important system performance measures are derived. The expression for expected total cost is also obtained. One can extend the present study to another one with Markovian Arrival Process (MAP) and positive service time.

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