ON CAYLEY FUNCTIONS THAT COMMUTE WITH IDEMPOTENTS

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Abstract: A Cayley function is a function α: S→S such that there exist an associative binary operation α: S × S → S and an element a ∈ S such that α(x) = a * x ∀x ∈ S. All Cayley functions α have been characterized algebraically (in powers of α) and also using directed graphs. We describe the Cayley idempotents that commute with any transformation. We also study the directed graph of functions that commute with an idempotent.

Keywords: Semigroups, Directed Graphs, Cayley Functions, Inner Translations.

Introduction: Consider a binary operation on $S = \{a, b, c, d\}$, both the rows and columns of the cayley table can be viewed as full transformations on S. In fact, one may check that every row commutes with every column. Saying that the row of $(S,*)$ commute with its columns is just another way of saying that $(S,*)$ is a semigroup. Prompted by this approach J. Araujo et al. [1] has described an approach to study semigroups. The solution to the approach (if found) would give a way to construct the cayley table of a semigroup from a Cayley function.

A Cayley function is a function $\alpha: S \to S$ such that there exist an associative binary operation $*: S \times S \to S$ and an element $a \in S$ such that $\alpha(x) = *(a, x)\ \forall x \in S$. In other words we can say that Cayley functions are inner translations of some semigroup. The characterizationAll Cayley functions have been characterized algebraically (in powers of α) by Zupnik [4]. A characterization (using directed graphs) of Cayley functions has been done by Araujo,J. [1].

Let S be a semigroup. For a fixed $a \in S$, the mapping $\lambda_a : S \to S$ $[\rho_a : S \to S]$ defined by $\lambda_a(x) = ax$ $[\rho_a(x) = xa]$ is called a left [right] inner translation of S. If S is a finite group, then $\lambda_a$is a regular permutation on S, that is, λ_a is a product of disjoint cycles of the same length. Let α be a transformation on a set S. We say that α is a Cayley function on S if there is a semigroup with universe S such that α is an inner translation of the semigroup S. Note that α is a left inner translation of a semigroup $(S,*)$ if and only if α is a right inner translation of the semigroup $(S,.)$, where for all $a, b \in S$, $a * b = b.a$.

For the reminder of this paper, we fix a non empty set S and denote $T(S)$ the set of all transformations on S, α will denote a function on S. For any positive integer n, α^ndenotes the n^{th}iterate of α, α^0denotes identity function on S. Suppose there is an integer s such that range $img(\alpha^n) = img(\alpha^{n+1})$ if and only if $n \geq s$; then s will be called the (range)stabilizer of α .

Preliminaries : A directed graph is an ordered pair $D = (S, \rho)$ where the elements of S are called vertices and A is a set of ordered pairs of vertices (binary relation), called arrows, directed edges. Any pair $(a, b) \in \rho$ is called an arc of D, which we will write as $a \to b$. A vertex a is called is called an initial vertex in D if there is no b in ρ such that $b \to a$; it is called a terminal vertex in D if there is no $b \in S$ such that $a \to b$

Definition 1: *A digraph $D$is called a functional digraph if there is $\alpha : S \to S$such that for all $x, y \in S, x \to y$is an arc in Dif and only if $\alpha(x) = y$.*

Such a functional digraph is denoted as $D_\alpha$as there is one and only one function that represents a functional di-graph. Let D be a digraph and if there exists pairwise distinct vertices $\ldots x_{-2} x_{-1}, x_0, x_1, \ldots$ of D such that $x_0 \to x_1 \to \ldots\ldots \to x_{k-1} \to x_0$then the graph is said to have a **cycle** of length k denoted as $(x_0 x_1 \dots x_{k-1})$. D has a **chain** of length m, denoted $[x_0 x_1 \dots x_m]$ if there are pairwise distinct vertices such that $x_0 \to x_1 \to \dots\dots \to x_m$ in D. Similarly D has a **right ray** [**left ray or double ray**] denoted $[x_0\, x_1 x_2 \dots\rangle$; $[\langle \dots x_2 x_1 x_0], \langle \dots x_{-1} x_0\, x_1 \dots\rangle]$ if there exist pairwise distinct vertices such that $x_0 \to x_1 \to x_0 \to \dots\dots [(\dots\dots \to x_2 \to x_1 \to x_0)$ or $(\dots \to x_{-1} \to x_0 \to x_1 \to \dots)]$. Let S be a non-empty set, then $T(S)$ denoted the set of all functions on S, $\alpha \in T(S)$ (i.e, α is a function on S) and D_α the directed graph that represents α, then a right ray $[x_0\, x_1 x_2 \dots\rangle$ in D_α is called a **maximal right ray** if x_0 is an initial vertex of D_α.

Definition 2: *Let D_α be a functional digraph, where $\alpha \in T(S)$*

- *A left ray $L = \langle \dots x_2 x_1 x_0]$ in D_α is called an **infinite branch** of a cycle C[double ray W] in D_α if x_0 lies on $C[W]$ and x_1 does not lie on $C[W]$. We will refer to any such L as an infinite branch in D_α.*
- *A chain $P = [x_0 x_1 \dots x_m]$ of length $m \geq 1$ in D_α is called a **finite branch** of a cycle C[double ray W, maximal right ray R, infinite branch L] in D_α if x_0 is an initial vertex of D_α, x_m lies on C$[W, R, L]$ and $x_m \geq 1$ does not lie on $C[W, R, L]$. If x_m lies on an infinite branch $L = \langle \dots y_2 y_1 y_0]$, we also require that $x_m \neq y_0$.*

The components of D_α correspond to the connected components of the underlying undirected graph of D_α where the component containing x is defined as follows

Definition 3: *Let $\alpha \in T(S)$, $x \in S$. The subgraph of D_α induced by the set $\{y \in S : \alpha^k(y) = \alpha^m(x)$ for some integers $k, m \geq 0\}$ is called the component of D_α containing x.*

The following proposition is due to [3] shows the character of functional digraphs.

Proposition 1:*Let D_α be a functional digraph. Then for every component A of :*

(1) if A has a (unique) cycle C, then A is the join of C and its branches;
(2) if A has a double ray W, then A is the join of W and its branches;
(3) if A has a maximal right ray R but not a double ray, then A is the join of R and its (finite) branches(type rro*).*

We now look at some properties functional digraphs representing transformations that have the stabilizer.

Definition 4: *Let $\alpha \in T(S)$. The stable image of* α*, denote* sim(α)*, is a subset of* S *defined by* If $\alpha \in T(S)$, then: $sim(\alpha) = \{x \in S : x \in img(\alpha^n)$ $for\ every\ n \geq 0\}$

$sim(\alpha)$ consists of the vertices of D_α that lie on cycles, double rays, or infinite branches; $sim(\alpha) = \varphi$ if and only if each component of D_α is of type rro.

Definition 5: *The stabilizer of $\alpha \in T(S)$ as the smallest integer $s \geq 0$ such that $img(\alpha^s) = img(\alpha^{s+1})$. If such a s does not exist, we say that α has no stabilizer.* If $\alpha \in T(S)$, then:

- the stabilizer of α is the smallest integer $s \geq 0$ such that $\alpha^s(x) \in sim(\alpha)$ for every $x \in S$;
- has the stabilizer $s = 0$ if and only if $img(\alpha) = sim(\alpha) = S$, which happens if and only if each component A of D_α is either the join of a cycle C and the infinite branches of C or the join of a double ray W and the infinite branches of W ;
- if α has the stabilizer s, then $sim(\alpha) = img(\alpha^s)$. A transformation may have a non-empty stable image and no stabilizer.

Definition 6: *Let $\alpha \in T(S)$. A finite branch $[x_0 x_1 \dots x_m]$ in D_α is called a twig in D_α if $x_m \in sim(\alpha)$ (that is, x_m lies on a cycle, double ray, or infinite branch) and $x_p \notin sim(\alpha)$ for every $P \in \{0, \dots, m-1\}$.* Not every finite branch is a twig.

Cayley Functions: The following definition is due to Zupnik[4], though there is slight variation due brought by Araujo in [1]

Definition 7: *Suppose $\alpha \in T(S)$ has the stabilizer s. If s >0, we define the subset Ω_α of S by: $\Omega_\alpha = \{a \in S: \alpha^s(a) \in sim(\alpha)$* but $\alpha^{s-1}(a) \notin sim(\alpha)\}$ *If $s = 0$, we define Ω_α to be S.* If $\alpha \in T(S)$ has the stabilizer

$s > 0$, then $\Omega_\alpha$consists of the initial vertices of the twigs of length s in D_α.

Theorem 1:*Let $\alpha \in T(S)$. Then $\alpha$is a Cayley function if and only if exactly one of the following conditions holds:*

a) *has no stabilizer and there exists $a \in S$such that $\alpha^n(a) \notin img(\alpha^{n+1})$for every $n \geq 0$;*

b) *has the stabilizer s such that $\alpha|img(\alpha^s)$is one-to-one and there exists $a \in \Omega_\alpha$such that $\alpha^m(a) = \alpha^n(a)$implies $\alpha^m = \alpha^n$for all $m, n \geq 0$; or*

c) *has the stabilizer ssuch that $\alpha|img(\alpha^s)$is not one-to-one and there exists $a \in \Omega_\alpha$ such that:*
   (1) $\alpha^m(a) = \alpha^n(a)$implies $m = n$for all $m, n \geq 0$; and
   (2) For every $n > s$, there are pairwise distinct elements $y_1, y_2, \ldots$of S such that $\alpha(y_1) = \alpha^n(a)$, $\alpha(y_k) = y_{k-1}$for every $k \geq 2$, and if $n > 0$ then $y_1 \neq \alpha^{n-1}$(a)

We have as a corollary that all idempotent functions are Cayley functions.

Corollary 1: *All idempotent functions are Cayley functions.*
Proof:Suppose ε is an idempotent function from S to S then $\varepsilon * \varepsilon(x) = \varepsilon(x)$ for all $x \in S$. So we have that the stabilizer of ε is 1 ($s = 1$) for all idempotent other than identity. Now from the above theorem (b condition) we can say that ε is a Cayley function, as for any a ∈ Ω_ε we can that $\varepsilon^m(a) = \varepsilon^n(a)$ implies $\varepsilon^m = \varepsilon^n$for all m, n ≥ 0.
The following three theorems characterize the directed graph of the Cayley functions

Theorem 2: [1] *Let $\alpha \in T(S)$be such that $D_\alpha$has a component of type rro. Then $\alpha$is a Cayley function if and only if $D_\alpha$has a component of type rro such that:*

1. *A is the join of maximal right ray $[x_0x_1x_2 \ldots\rangle$and its branches;*
2. *for every $i \geq 1$, if $[y_0y_1 \ldots y_m = x_i]$is a branch of R, then $m \leq i$.*

Theorem 3: [1] *Let $\alpha \in T(S)$be such that every component of $D_\alpha$has a unique cycle or a double ray, and $D_\alpha$does not have an infinite branch. Then $\alpha$is a Cayley function if and only if the following conditions are satisfied:*

1. s = $sup_b(\alpha)$ *is finite;*
2. *if $s > 0$ and D_α has a double ray, then some double ray in* D_α *has a branch of length* s
3. *if $D_\alpha$does not have a double ray, then there are integers $1 \leq k_1 \leq k_2 \leq \ldots \leq k_p$, $p \geq 1$, such that*
   a. *$\{k_1, \ldots, k_p\}$ is the set of the lengths of the cycles in* D_α*;*
   b. k_i *divides* k_p *for every $i \in \{1, \ldots, p\}$and*
   c. *if $s > 0$, then some cycle of $D_\alpha$of length $k_p$has a branch of length s.*

Theorem 4: [1] *Let $\alpha \in T(S)$be such that every component of $D_\alpha$has a unique cycle or a double ray, and $D_\alpha$has an infinite branch. Then $\alpha$is a Cayley function if and only if the following conditions are satisfied:*

1. *$s = sup_t(\alpha)$is finite;*
2. *$D_\alpha$has a double ray $W = \langle \ldots x_{-1}x_0x_1 \ldots \rangle$such that for some x_i:*
   a. if $s > 0$ then W *has a finite branch at xi of lengths;*
   b. W has an infinite branch at each $x_j$with $j > i$.

Functions That Commute with Cayley Idempotents: Let S be a non-empty set. For a transformation $\alpha \in T(X)$, the centralizer $C(\alpha)$ of a transformation α on a set S is the set of all elements in $T(S)$ that commute with α i.e, $C(\alpha) = \{\beta \in T(S) : \beta\alpha = \alpha\beta\}$ All idempotent functions are Cayley functions from Corollary 1.

Proposition 2:[2] *Let $\alpha, \beta \in T(S)$. Then $\beta \in C(\alpha)$if and only if for every connected component γ of α, there exists a connected component δ of α such that $\beta|dom(\gamma)$is a graph homomorphism from $D(\gamma)$to $D(\delta)$.*

The centralizers in the full transformation semigroup have been studied in [2]. Now as a partial solution to Problem 1 stated by Arau´jo, et all in [1] we have the following.

Theorem 5: *Let $\alpha \in T(X)$, then the Cayley idempotents that commute with α are exactly all the idempotents εthat commute with α i.e,* $\{\varepsilon : \varepsilon \in C(\alpha)\}$

Proof:From Corollary 1 we have that all idempotents are Cayley functions and hence Cayley Idempotents. Therefore the idempotents that commute with α are the Cayley Idempotents that commute with α.

From corollary 5.2 of [2] we have

Theorem 6: *Let $\varepsilon, \alpha \in T(X)$, where ε is an idempotent. Then α ∈ C(ε) if and only if for every connected component γ of ε with cycle y, there exists a connected component δ of ε with cycle z such that $y\alpha = z$and $(dom(\gamma))\alpha \subseteq dom(\delta)$.*

Let $\varepsilon \in T(X)$ and let $C_\varepsilon$be the set of connected components of ε. Each component of ε will have a one cycle and may have branches of length one or may not have any branch at all but at least one component has a branch of length one as the stabilizer of ε is 1.

Now for $\alpha \in C(\varepsilon)$, we can define a function $\Phi_\alpha$on $C_\varepsilon$by

$\Phi_\alpha(\gamma)$ = the unique $\delta \in C_\varepsilon$such that$(dom(\gamma))\alpha \subseteq dom(\delta)$

$\Phi_\alpha$is well defined from theorem 6 and the following lemma follows immediately from the definition of $\Phi_\alpha$and theorem 6.

Lemma 1: *Let $\varepsilon \in T(X)$be an idempotent and $\alpha \in C(\varepsilon)$. Suppose that A is a connected component of $D\Phi_\alpha$with cycle $(\gamma_0 \ldots \gamma_{k-1})$, and that Zis the set of all elements $x \in S$such that xis in some $\gamma \in A$. Then*

1. *if $\gamma\ \delta \in A$is such that $\Phi_\alpha(\gamma) = \delta$, then for $x \in \gamma, \alpha(x)$is in δ;*
2. *for every $x \in Z$, $\alpha(x)$is in Z, that is $\alpha/_Z \in T(Z)$;*
3. *if $x \in Z$ is not in any γ_i, then x does not lie on any cycle of D_α.*

Lemma 2: *Let $\varepsilon \in T(X)$be an idempotent and $\alpha \in C(\varepsilon)$. Suppose that A is a connected component of $D\Phi_\alpha$with cycle $(\gamma_0 \ldots \gamma_{k-1})$, and that Zis the set of all elements $x \in S$such that xis in some $\gamma \in A$ and each $\gamma_i$has a 1-cycle x_i. Then*

1. $\alpha^k(x_0) = x_0$
2. *then αhas at least one cycle of length k*
3. *every cycle of $D_{\alpha/Z}$ has length m.k (i.e. a multiple of k)*

Proof:Since $(\gamma_0 \ldots \gamma_{k-1})$is a cycle in , $\Phi^k{}_\alpha(\gamma_0) = \gamma_0$and as $\alpha \in C(\varepsilon)$ cycles will be mapped to cycles proving (1) and hence the corresponding 1-cycle $x_i$of $\gamma_i$forms a cycle of length k proving (2). Suppose $x_i^1$is a branch of length one in $\gamma_i$and x_i^1belongs to some t-cycle in α (i.e.$\alpha^t(x_i^1) = x_i^1$) then $\Phi^t{}_\alpha(\gamma_i) = \gamma_i$ but as $\Phi^k{}_\alpha(\gamma_i) = \gamma_i$we have that $t = km$.

Lemma 3: *Let $\varepsilon \in T(X)$be an idempotent and $\alpha \in C(\varepsilon)$. Suppose that A is a connected component of $D\Phi_\alpha$with cycle $(\gamma_0 \ldots \gamma_{k-1})$, and that Zis the set of all elements $x \in S$such that xis in some $\gamma \in A$and each $\gamma_i$has a 1-cycle x_i. Assume that* s *is the length of a branch in A(s = 0 if Ahas no branches), if $s \leq 1$then there exists a branch of length s in $D\alpha|Z$*

Proof:Let $[\delta_0\delta_1 \ldots \delta_m = \gamma_i]$ be a branch of A, $x_{\delta_i}$the 1-cycle corresponding to each $\delta_i$then $[x_{\delta_0} \ldots x_{\delta_m} = x_i]$ will be a branch on the cycle $(x_0 \ldots\ldots x_{k-1})$.

Lemma 4: *Let $\varepsilon \in T(X)$be an idempotent and $\alpha \in C(\varepsilon)$. Suppose that A is a connected component of $D\Phi_\alpha$with cycle $(\gamma_0 \ldots \gamma_{k-1})$, and that Zis the set of all elements $x \in S$ such that xis in some $\gamma \in A$ and each $\gamma_i$has a 1-cycle $x_i$and branches $x_i^1, x_i^2, \ldots$. Then the maximum length of cycles in αis m.k where* m *is the* min $l_1, \ldots. l_{k-1}$*where $l_i$is the number of branches in γ_i.*

Though the maximum length can be identified there may or may not be a cycle of that length. If we further suppose that within the cycle$(\gamma_0 \ldots \gamma_{k-1})$ at least one cycle say $x_i$has no branches then we can say that precisely calculate the length of the cycle (which will be k).

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