UNIQUE MINIMUM EDGE DOMINATION IN TREES

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Abstract: For any graph G and any edge $e \in E(G)$ we define $N'(e) = \{f \in E(G) \mid f \text{ adjacent to } e\}$ and the set $N'[e] = N'(e) \cup \{e\}$. If $B \subseteq E(G)$, then $N'(B) = \cup \notin e \in B\ N'(e)$ and $N'[B] = N'(B) \cup B$. For a subset F of E(G) and an edge $e \in F$ we define the set $P'(e, F) = N'[e] \setminus N'[F \setminus \{e\}]$,
and we call an edge $f \in P'(e, F)$ a private adjacent edge of e with regard to F. Further, we give a characterization of trees with unique minimum edge dominating sets.

Keywords: Edge dominating set, unique minimum edge dominating set and component.

Introduction:
Lemma: Let G be a connected graph of order at least 3 and let F be a unique minimum edge dominating set of G. Then the set F is independent, and every edge $e \in F$ contains at least two non adjacent edges in $P'(e, F)$.
Proof: Let $e \in F$ be arbitrary. Since F is minimal, we have

$$P'(e, F) \neq \emptyset.$$

If $P'(e, F) = \{e\}$, then we can take any edge f adjacent to e and

$$(F \setminus \{e\}) \cup \{f\}$$

Is a minimum edge dominating set of G different from F, which is a contradiction.
If

$$f \in P'(e, F) \setminus \{e\} \neq \emptyset$$

and every edge in $P'(e, F) \setminus \{f\}$ is adjacent to f, then again

$$(F \setminus \{e\}) \cup \{f\}$$

is a minimum edge dominating set of G different from F, which is a contradiction.

Hence, for every edge $e \in F$ the set $P'(e, F)$ contains two non adjacent edges. This also implies that no two edges in F are adjacent.

The next theorem is a characterization of trees with unique minimum edge dominating sets similar to the characterization in Corollary. One part of this theorem says that, for trees, the necessary condition in Lemma is also sufficient. The converse does not hold in general, as we can see with the simple graph G with vertex set

$$V(G) = \{u, v, w, x\},$$

edge set

$$E(G) = \{uv, uw, ux, vw\}$$

and with the two minimum edge dominating sets {uv} and {uw}.

Theorem: Let T be a tree of order at least 3 and let F be a subset of E(T).
Then the following conditions are equivalent:
1. F is the unique minimum edge dominating set of T.
2. F is an edge dominating set of T such that every edge e in F has at least two non adjacent edges in $P'(e, F)$.
3. F is an independent edge dominating set of T such that every edge e in F has at least two non adjacent edges in $P'(e, F)$.
4. F is a minimum edge dominating set of T such that $\gamma'(T - e) > \gamma'(T)$ for every edge $E \in F$.

Proof:

(i) ⇒ (iii):

Follows immediately from Lemma.

(iii) ⇒ (ii):

Obviously.

(ii) ⇒ (i):

Let F be an edge dominating set of T as in (ii).

For any subset B of the edge set of T we define

$$V(B) = \{u, u' \in V(T) \mid uu' \in B\}.$$

Thus, for every edge $e = vw \in F$ there are two edges vv' and ww' with

$$v' \neq w' \text{ and } v, v', w, w' \notin V(F \setminus \{e\}).$$

Hence, no two edges in F are adjacent. Suppose there is a minimum edge dominating set $F' \neq F$ of T.

Then,

$$|F \setminus F'| \geq |F' \setminus F|.$$

Let the set

$$B = (F \setminus F') \cup (F' \setminus F)$$

and

$$H = T[V(B)].$$

Let

$$F_{1'} = \{vw \in F' \setminus F \mid v, w \in V(F \setminus F')\},$$

$$F_{2'} = \{vw \in F' \setminus F \mid |\{v,w\} \cap V(F \setminus F')| = 1\},$$

and

$$F_{3'} = \{vw \in F' \setminus F \mid v, w \notin V(F \setminus F')\}.$$

The set $F' \setminus F$ is the disjoint union of $F_{1'}$, $F_{2'}$ and $F_{3'}$. We get for the vertex set of H

$$|V(H)| = |V(B)| \leq 2|F \setminus F'| + |F_{2'}| + 2|F_{3'}|.$$

By (ii), for every vertex $v \in V(F \setminus F')$ there is an edge $vw \in F \setminus F'$ and an edge $vv' \neq vw$ such that $v, v' \notin V(F \setminus \{e\})$.

Since F' is an edge dominating set of T, we get that v or v' is in $V(F')$.

If

$$v \in (V(F) \setminus V(F')) \subseteq V(F \setminus F'),$$

then

$$v' \in (V(F') \setminus V(F)) \subseteq V(F' \setminus F)$$

and

$$vv' \in E(H) \setminus B.$$

This implies that

$$|E(H) \setminus B| \geq |V(F) \setminus V(F')| \geq 2|F \setminus F'| - 2|F_{1'}| - |F_{2'}|.$$

Hence, we obtain for the cardinality of E(H)

$$\begin{aligned}|E(H)| &= |F \setminus F'| + |F' \setminus F| + |E(H) \setminus B| \\ &\geq 2|F' \setminus F| + (2|F \setminus F'| - 2|F_{1'}| - |F_{2'}|) \\ &= 2(|F_{1'}| + |F_{2'}| + |F_{3'}|) + (2|F \setminus F'| - 2|F_{1'}| - |F_{2'}|) \\ &= |F_{2'}| + 2|F_{3'}| + 2|F \setminus F'| \\ &\geq |V(H)|.\end{aligned}$$

But, since H is a forest, we have

$m(H) = n(H) - \kappa(H) < n(H)$, which is a contradiction.

(i) ⇒ (iv):

Let F be the unique minimum edge dominating set of T,

let

$$e = v_1v_2 \in F$$

be arbitrary, and let T_1 and T_2 be the components of $T - e$ where $v_i \in V(T_i)$ forest $i = 1, 2$.

Further, let F' be a minimum edge dominating set of $T - e$ and for $i = 1, 2$

let

$$F_i = F \cap E(T_i)$$

and

$$F_{i'} = F' \cap E(T_i).$$

By (i) ⇒ (ii), the edge e is adjacent to at least two edges $v_1w_1 \in E(T_1)$ and $v_2w_2 \in E(T_2)$ that are not adjacent to any other edge in F. Hence, the set F_i is not an edge dominating set of Ti, contrary to $F_{i'}$ for $i \in \{1, 2\}$.

Thus, we have $F_i \neq F_{i'}$ for i = 1, 2.

Since the set

$$F_{i''} = (F \setminus F_i) \cup F_{i'} \neq F$$

is an edge dominating set of T, we get $|F_i| < |F_{i'}|$.

This yields

$$\gamma' (T - e) = |F'| = |F_{1'}| + |F_{2'}| \geq |F_1| + |F_2| + 2 = |F| + 1 > \gamma' (T).$$

(iv) ⇒ (i):

Let F be a minimum edge dominating set of T such that

$$\gamma' (T - e) > \gamma' (T)$$

for every edge $e \in F$.

Suppose that there is a minimum edge dominating set $F' \neq F$ of T. There exists at least one edge $e \in F \setminus F'$ and the set F′ is an edge dominating set of T –e.

Hence,

$$\gamma' (T - e) \leq |F'| = \gamma' (T)$$

for some $e \in F$, which is a contradiction.

Corollary: Let T be a tree of diameter at least 3, let F be a minimum edge dominating set of T, and let $e \in F$ arbitrary. Then F is the unique minimum edge dominating set of T if and only if every component of the forest H = T –N_[e] is of order at least 4 and H has the unique minimum edge dominating set $F \setminus \{e\}$.

Proof: Let F be a minimum edge dominating set of T and let $e \in F$ be arbitrary.

First, let F be unique.

Hence F fulfils (ii) in Theorem, and this implies that $F \setminus \{e\}$ fulfils (ii) for the forest H.

Thus each component of H is of order at least 4. If we use Theorem on these components, then we get that H has the unique minimum edge dominating set $F \setminus \{e\}$.

Now, let $F \setminus \{e\}$ be the unique edge dominating set of H and every component of H be of order at least 4. By Theorem, the set $F \setminus \{e\}$ fulfils (ii). This implies that F fulfils (ii) for T.

Thus F is unique, by Theorem.

Conclusion: We present a linear time algorithm for finding a minimum edge dominating set of a block graph. Finally we given a characterization of trees with unique minimum distance domination sets, and trees with unique minimum edge domination sets.

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