EFFECT OF MIGRATION OF SUSCEPTIBLE PREY IN ECO-EPIDEMIOLOGICAL SYSTEM: A MATHEMATICAL APPROACH

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Abstract: In a two-patch environment, a predator-prey system is considered where the prey population is divided into two parts: susceptible prey and infected prey. Here we consider the migration of the susceptible prey and predator between those patches. Predator-prey system with infection in the susceptible prey is well studied but the analysis of this system with migration of the susceptible prey and the predator is not yet been explored. Here we take the advantage of two different time scales and apply the aggregation procedure to acquire the aggregated system. Here we assume migration of the predator is constant, whereas the migration of the prey is dependent on the predator density. We wish to observe the migratory effect of susceptible prey through stability of our proposed slow-fast system. Our results depict that migration of the susceptible prey can stabilize the system within the two-patch environment.

Keywords: Migration, Predator, Prey, Two-patch environment, Slow-fast system.

Introduction: Eco-epidemiology is one of foremost arena of study that compacts with both ecological and epidemiological avenues. The environment plays a major responsibility for migration of prey-predator. In the epidemiological aspect, a lot of mathematical work has been done but the pioneering work of epidemiological model, done by Anderson and May [1], added a new dimension for growing research in this field. In the last few decades many researchers devoted their time to study the effects of disease on predator-prey system [2], [3]. Recently most of the research works under mathematical eco-epidemiology has been given a considerable attention for spraying of the disease towards prey population [2], [3], [4].

As far as our knowledge concerned, the effect of migration on predator-prey model with infection in prey population have not yet been explored explicitly, though we believe that this study may open a new research arena. There are two types of migration; one is density dependent and other is density independent migration. Abdllaoui etal., [5] studied a predator-prey system in presence of density dependent migration on two patch predator-prey system in the environment.

In most of the mathematical models of population dynamics in the region of eco-epidemiology, it is considered that environments are homogeneous [6], [7]. But in reality, the environment is not generally regular or steady in nature but also patchy [8]. Within various patches of the environment, migration is one of the natural phenomenon from one patch into other [5] either for survival issue or some other environmental causes.

We consider here that migration of prey and predator are faster rather than the growth and their interaction within one patch. This is because; we apply two different time scales in our model dynamics. Slow-fast mechanisms are better procedure as well as techniques to analyze the population dynamics of this type of complex two-patch system.Aggregation methods are applied to reduce a large and complex system of many variables into a simple system of few variables. So we can take the benefit of two time scales to govern the mathematical model analytically by aggregating the variables, which can be easily handled rather than the original model. Many researchers devoted their time to these methods [9], [10].

In this work we study a two patch predator-prey system where prey is divided into two classes one is susceptible and another one is infected. The patches are connected by migration of the susceptible prey and predator. The susceptible class grows logistically. Then we develop a slow-fast model. In this research article, we wish to obtain the benefit of two various time scales and apply the aggregation mechanism to attain the modified system of population dynamics within the environment. We desire to notice the effect of migration for susceptible prey of our proposed mathematical model, connected with the slow-fast system. Actually, our aim is to search out, if there is any effect of migration that can have the ability to stabilize the system dynamics with the help of slow-fast system within the two-patch environment.

Development of the Mathematical Model

We consider first the predator-prey model, where the prey population is divided into susceptible and infected class.

$$\frac{dS}{dt} = r_1 S\left(1 - \frac{S}{k_1}\right) - \alpha_1 SP - \lambda SI,$$
$$\frac{dI}{dt} = \lambda SI - \alpha_2 IP - \mu_1 I \qquad (1)$$
$$\frac{dP}{dt} = c_1\alpha_1 SP + c_2\alpha_2 IP - \mu_2 P$$

where S, I and Pare the densities of susceptible prey, infected prey and predator population respectively. The intrinsic growth rate of the susceptible prey is given by r_1 and k_1 is the carrying capacity of the susceptible prey. The searching rate of the susceptible prey and the infected prey are denoted by α_1 and α_2 respectively. The infection rate is considered as λ. The death rate of the infected prey and the predator population are assumed by μ_1 and μ_2 respectively. Finally c_1 is the conversion efficiency of the susceptible prey and c_2 is the conversion factor of the infected prey population.

Now we incorporate the concept of migration in the system of equation (1). Here we introduce the existence of two patches in the environment. In both the patches, the susceptible prey population grows logistically. Whenever the predator migrates into patch I for food, the susceptible prey migrates to patch II to avoid predation and vice versa. Since the infected prey is not healthy enough to move, they are not able to migrate. Due to the scarcity of food in patch I, the predator population migrates in to patch II within the environment.

Our mathematical model is the composition of two major constituents of the eco-epidemiological system. The first component terms the susceptible prey (S), infected prey (I) and predator (P) growth and their interactions on each patch, while the second component defines the dispersal through migration of susceptible prey and the predator.

We assume that the migration rate of the susceptible prey between two patches is very high and the corresponding migration of the predator is also very high. As a result, the change of the density of the susceptible prey and predator due to migration are much higher compared to the change in concentration due to birth, death and interactions with prey and predators. We also assume that the change of the concentration of the prey population due to infection is low. Therefore two different time scales have been considered here: the fast one corresponds to migration and the second one is due to birth, death, interactions and infection of prey population.

Let S_1, I and P_1 are the densities of the susceptible prey, infected prey and the predator population in patch I respectively and S_2 and P_2 are the densities of the susceptible prey and predator population respectively in patch II. We have considered that migration of the predator population is constant, whereas migration of the susceptible prey is predator dependent. The susceptible prey migrates into patch II, when predator density is large in patch I and migrates from patch II, when the predator density is very high in patch II.

Thus the system of equation (1) can be written as follows:

$$\begin{aligned}
\frac{dS_1}{dt} &= M_2S_2P_2 - M_1S_1P_1 + \varepsilon\left[r_1S_1\left(1-\frac{S_1}{k_1}\right) - \alpha_1S_1P_1 - \lambda S_1 I\right],\\
\frac{dS_2}{dt} &= M_1S_1P_1 - M_2S_2P_2 + \varepsilon\left[r_2S_2\left(1-\frac{S_2}{k_2}\right) - \alpha_3S_2P_2\right],\\
\frac{dI}{dt} &= \varepsilon\left[\lambda S_1 I - \alpha_2 IP_1 - \mu_1 I\right], \qquad (2)\\
\frac{dP_1}{dt} &= F_2P_2 - F_1P_1 + \varepsilon\left[c_1\alpha_1S_1P_1 + c_2\alpha_2IP_1 - \mu_2P_1\right],\\
\frac{dP_2}{dt} &= F_1P_1 - F_2P_2 + \varepsilon\left[c_3\alpha_3S_2P_2 - \mu_2P_2\right],
\end{aligned}$$

where M_1 and M_2 are the migration rates of prey population from patch I into patch II and vice versa which is predator density dependent. Also F_1 and F_2 are the migration rates of predator population from patch I and patchII respectively. The system has to be analyzed with the initial conditions $S_1(0) = S_{10} > 0$, $I(0) = I_0 > 0$, $S_2(0) = S_{20} > 0$, $P_1(0) = P_{10} > 0$ and $P_2(0) = P_{20} > 0$. Now, we perform perturbation technique to aggregate variables corresponding to the asymptotically stable fast equilibrium point and obtain a global (or aggregated) model easier to handle (at slow timescale), which approximates the initial one [9], [10].

Aggregation of the Mathematical Model: We observe that, the system (2) is mainly determined by the migration part, the demographic one is being only a small perturbation. We are now very much interested in the fast dynamics and the corresponding fast model is obtained by neglecting the slow part of the population dynamics i.e., taking $\varepsilon = 0$,

$$\frac{dS_1}{dt} = M_2S_2P_2 - M_1S_1P_1, \frac{dS_2}{dt} = M_1S_1P_1 - M_2S_2P_2,$$
$$\frac{dP_1}{dt} = F_2P_2 - F_1P_1, \frac{dP_2}{dt} = F_1P_1 - F_2P_2. \qquad (3)$$

Let us indicate $S_1 + S_2 = S$ and $P_1 + P_2 = P$, which are the total population densities. These are the constants of motion of the system (3). The fast equilibrium point is the solution of the following system of equations:

$$M_2S_2P_2 - M_1S_1P_1 = 0, S_1 + S_2 = S,$$
$$F_2P_2 - F_1P_1 = 0, P_1 + P_2 = P, \qquad (4)$$

Which is equivalent to

$$S_1 = \frac{FS}{M+F}, \quad S_2 = \frac{MS}{M+F}, \quad P_1 = \frac{P}{F+1}, \quad P_2 = \frac{FP}{F+1}, \qquad (5)$$

where $M = \frac{M_1}{M_2}$ and $F = \frac{F_1}{F_2}$, are the migration ratio of the susceptible prey and predator respectively. Now,

$$\frac{dS_1}{dt} = M_2S_2P_2 - M_1S_1P_1 = M_2P_2(S - S_1) - M_1S_1P_1 = G_1(S_1) \text{ (say)}. \qquad (6)$$

Then, $G'_1(S_1) = -(M_2P_2 + M_1P_1) < 0$. Similarly $G'_2(P_1) = -(F_1 + F_2) < 0$. Therefore the fast equilibrium is always asymptotically stable. Now we are aggregating the variables corresponding to the same population as $S_1 + S_2 = S$, $I = i$ and $P_1 + P_2 = P$. The global model at slow time scale is obtained as:

$$\frac{dS}{d\tau} = RS\left(1-\frac{S}{k}\right) - \alpha SP - \beta Si,$$
$$\frac{di}{d\tau} = \beta Si - \alpha_2 p_1 iP - \mu_1 i, \qquad (7)$$
$$\frac{dP}{d\tau} = D_1 SP + D_2 iP - \mu_2 P,$$

where, $R = r_1 f_1 + r_2 f_2$, $k = (r_1 f_1 + r_2 f_2)/(\frac{r_1 f_1^2}{k_1} + \frac{r_2 f_2^2}{k_2})$, $\alpha = \alpha_1 f_1 p_1 + \alpha_3 f_2 p_2$, $\beta = \lambda f_1$, $D_1 = c_1\alpha_1 f_1 p_1 + c_3\alpha_3 f_2 p_2$, $D_2 = c_2\alpha_2 p_1$, $f_1 = \frac{F}{M+F}$, $f_2 = \frac{M}{M+F}$, $p_1 = \frac{1}{F+1}$, $p_2 = \frac{F}{F+1}$, $\tau = \varepsilon t$ (slow time scale).

Existence and the Local Stability Analysis of the Equilibrium Points: There are five equilibrium points of the aggregated system (7), trivial equilibrium point $E_0(0,0,0)$, axialequilibrium point $E_1(k,0,0)$, planer equilibrium point$E_2(S_2, i_2, 0)$, $E_3(S_3, 0, P_3)$and the interior equilibrium point$E^*(S^*, i^*, P^*)$, where $S_2 = \frac{\mu_1}{\beta}$, $i_2 = \frac{R}{\beta}(1-\frac{S_2}{k})$, $S_3 = \frac{\mu_2}{D_1}$, $P_3 = \frac{R}{\alpha}(1-\frac{S_3}{k})$,

$$S^* = \frac{(R+\frac{\alpha\mu_1}{\alpha_2 p_1}-\frac{\beta\mu_2}{D_2})}{(\frac{R}{k}+\frac{\alpha\beta}{\alpha_2 p_1}-\frac{\beta D_1}{D_2})},\quad i^* = \frac{\mu_2 - D_1 S^*}{D_2} \text{ and } P^* = \frac{\beta S^* - \mu_1}{\alpha_2 p_1}.$$

It is obvious that the trivial equilibrium point E_0 and the axial equilibriumpoint E_1 always exist. The planer equilibrium points $E_2$and E_3exist if$k > S_2$ and $k > S_3$ respectively and the interior point equilibrium E^*existsif$S^* > 0$, $i^* > 0$ and $P^* > 0$. The local stability of the system (7) around each of the equilibria is obtained by computing the variational matrix corresponding to each equilibrium point the results thus obtained are stated below.

Lemma 1. The system (7) around $E_0(0,0,0)$ is always unstable.

Lemma 2. The system (7) around $E_0(k,0,0)$ is locally asymptotically stable (LAS) if $k < \min(\frac{\mu_1}{\beta}, \frac{\mu_2}{D_1})$.

Lemma 3. The system (7) around $E_2(S_2, i_2, 0)$ is LAS if$D_1 S_2 + D_2 i_2 < \mu_2$.

Lemma 4. The system (7) around$E_2(S_3, 0, P_3)$ is LAS if $\beta S_3 < \alpha_2 p_1 P_3 + \mu_1$.

Lemma 5. The system (7) around the interior point equilibrium $E^*(S^*, i^*, P^*)$is locally asymptotically stable if the roots of the characteristic equation$\xi^3 + A\xi^2 + B\xi + C = 0$ of the Jacobian matrix $J(E^*)$ satisfies the Routh-Hurwith criterion, i.e., $A > 0$, $C > 0$ and $AB - C > 0$,

$$\text{Where,} J(E^*) = \begin{bmatrix} A_1^* & A_2^* & A_3^* \\ B_1^* & 0 & B_3^* \\ C_1^* & C_2^* & 0 \end{bmatrix} \qquad (8)$$

and$A = -A_1^*$, $B = -(B_3^* C_2^* + A_2^* B_1^* + A_3^* C_1^*)$ and $C = A_1^* B_3^* C_2^* - A_2^* B_3^* C_1^* - A_3^* B_1^* C_2^*$ with $A_1^* = -\frac{RS^*}{k}$, $A_2^* = -\beta S^*$, $A_3^* = -\alpha S^*$, $B_1^* = \beta i^*$, $B_2^* = 0$,$B_3^* = -\alpha_3 p_1 i^*$, $C_1^* = D_1 P^*$, $C_2^* = D_2 P^*$, $C_3^* = 0$.

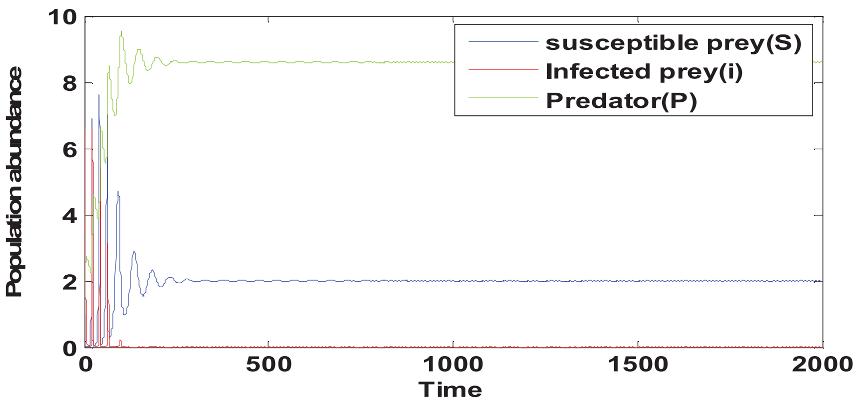

Figure 1: The figure depicts that infected population goes to extinction of the system (1) with r_1=0.9,k_1=45,μ_1=0.1,μ_2=0.02,λ=0.75,α_1=0.1,α_2=0.2,C_1=0.1,C_2=0.3.

Discussion with Numerical Outcome: In this section we perform the numerical experiments to observe the dynamics of the system (7) with the following set of parameter values. The parameter values are$r_1 = 0.9, k_1 = 45, r_2 = 0.5, k_2 = 45, \mu_1 = 0.1, \mu_2 = 0.02, \lambda = 0.75, \alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.1, C_1 = 0.1, C_2 = 0.3, C_3 = 0.1, M = 0.95, F = 1$.

These parameter values are kept fixed throughout the numerical experiments except the rate of infection (λ) and ratio of migration (M) of the susceptible prey. For the above set of parameter values we obtain the positive interior equilibrium$E^*(3.6568, 0.0556, 13.0948)$. For the same set of parameter values we have $A = 0.0289 > 0$, $C = 0.0011 > 0$ and $AB - C = 1.9221\, e^{-004} > 0$, which means that the system (7) is locally asymptotically stable around the positive interior equilibrium E^*.

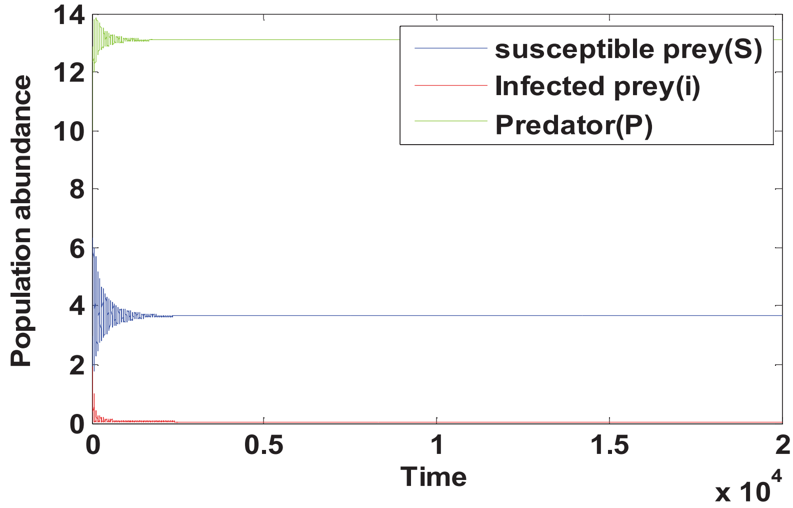

Figure 2: The figure depicts stable co-existence of three species of the system (7) with $r_1 = 0.9, k_1 = 45, r_2 = 0.5, k_2 = 45, \mu_1 = 0.1, \mu_2 = 0.02, \lambda = 0.75, \alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.1, C_1 = 0.1, C_2 = 0.3, C_3 = 0.1, M = 0.95, F = 1.$

2.

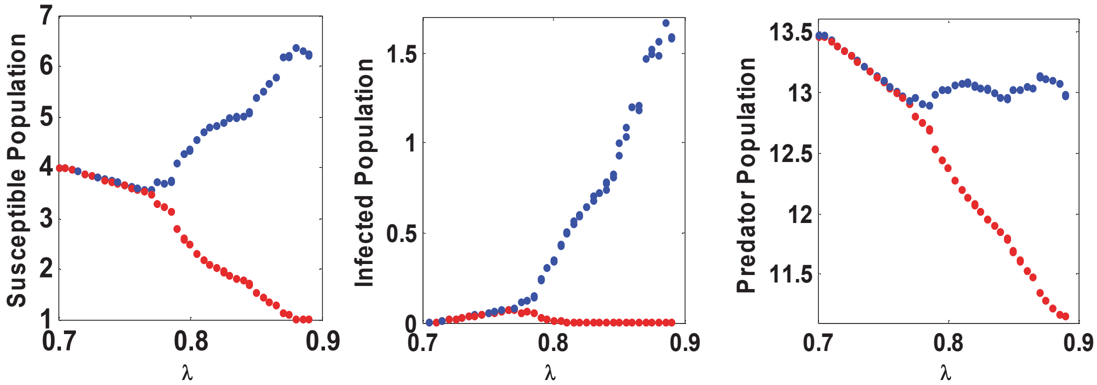

Figure 3:The bifurcation diagram of system (7) corresponds to the bifurcating parameter $\lambda \epsilon [0.7, 0.9]$ shows that the system enters into limit cycle oscillation from stability. Other parameter values are same as in Figure 2.

We observe that the infected population goes to extinction (Figure 1) in absence of migration whereas all the three population coexist (Figure 2) in presence of the migration of the susceptible prey and predator. When the infection rate increases (approximately from $\lambda = 0.77$), the system (7) shows limit cycle oscillation from steady state stable solution, which indicates that when the infection rate is very high, the system loose its stability. This scenario is shown by the bifurcating diagram (Figure 3).

To observe the effect of migration of the susceptible prey in the system (7), we fix the infection rate of the susceptible prey ($\lambda = 0.85$) and the migration ratio of the predator (F = 1). Furthermore, we vary the migration ratio of the susceptible prey (M) where the other parameter values are same as in Figure 2. Now for ($0.95 < M < 1.25$) system shows limit cycle oscillation, but again the increment of M, leads the system from limit cycle to stable steady state, which indicates that the migration of the susceptible prey have the potentiality to stabilize the system. To make it clear we capture the total scenario by the bifurcation diagram (Figure 4) for all the population.

Conclusion: Dynamical behavior of the system changed for different values of the infection rate (λ) of the susceptible prey, and the ratio of the migration

(M) of the susceptible prey. We observe that when the infection rate is very high, the system loses its stability and shows limit cycle oscillation (Figure 3).Further we observe that when the migration ratio of the susceptible prey is very high then again the system shows stability from limit cycle oscillation (Figure 4). Therefore, migration of the susceptible prey has a great impact on the dynamics of a predator-prey system of our newly formulating mathematical model.

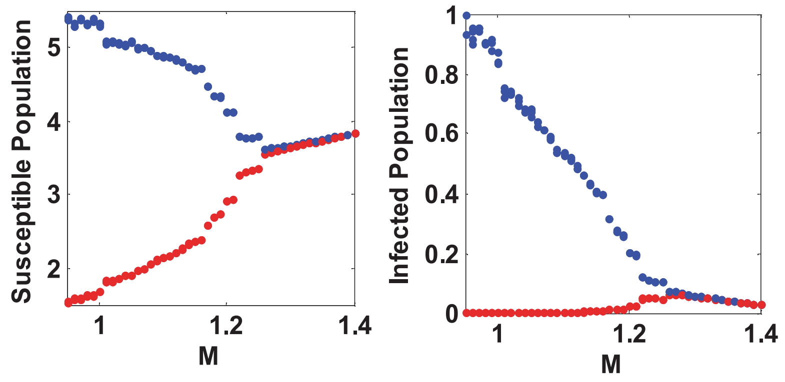

Figure 4: The bifurcation diagram of system (7) corresponds to the bifurcating parameter M ϵ[0.95, 1.4] shows that the system enters into stable steady state from limit cycle oscillation. Here λ = 0.85 and other parameter values are same as in Figure 2.

References:

1. Anderson, R. M., May, R., (1982). The invasion, persistence and spread of infectious diseases within animal and plant communities.Philos.Trans. R. Soc. Lond. Biol. Sci. 314, 533-570.
2. Beltrami, E., Coarrol, T., (1994). Modelling the role of viral disease in recurrent phytoplankton blooms. J. Math. Biol. 32, 857-863.
3. Hadeler, K. P., Freedman, H. I., (1989).Predator-prey populations with parasitic infection. J. Math. Biol. 27, 609-631.
4. Chattopadhyay, J., Arino, O., (1999).A predator-prey model with disease in the prey. Nonlinear Anal. 36, 747-766.
5. Abdllaoui, A. E., Auger, P., Kooi, B. W., Parra, R. B. de la, Mchich,R., (2007). Effects of density-dependent migrations on stability of a two patch predator-prey model. Mathematical Biosciences 210, 335-354.
6. Lotka, A. J. 1925. Elements of physical biology.Williams and Wilkins,Baltimore, Maryland, USA.
7. Beddington, J. R., Free, C. A., Lawton, J. H., (1975). Dynamic complexity in predator-prey models framed in difference equations. Nature.255, 58-60.
8. Emmel, T. C., (1976). Population biology. Harper and Row, New York, USA.
9. Auger, P., Poggiale, J.C., (1998). Aggregation and emergence in systems of ordinary differential equations, in: P. Antonelli, P. Auger (Eds.), Special Issue: Aggregation and Emergence in Population Dynamics, Math.Comput. Model.27, 1.
10. Auger, P., Parra, R. B. de la, Poggiale, J.C., Sanchez, E., Huu, T. N., (2007). Aggregation of variables and applications to population dynamics, Springer, Book chapter.

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