CHARACTERIZATION OF REGULAR* -T_i(i = 0,1,2) SPACES

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Abstract: In this paper, we introduce regular*-T_0, regular*-T_1, regular*-T_2 spaces using regular*-open sets and investigate their properties. Also we give characterizations for these spaces. Further we study relationship among themselves and with known separation axioms T_0, T_1, T_2 and semi-T_0, semi-T_1, semi-T_2.

Keywords: Regular*-T_0, regular*-T_1, regular*-T_2, regular*-open.

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Introduction: Separation axioms are useful in classifying topological spaces. Maheshwari.S.N. and Prasad.R.,[8] introduced the concept of semi-T_i(i=0,1,2) spaces using semi-open sets. Maki.H. Devi.R., and Balachandran.K.[9] introduced the concept of α-T_i(i=0,1,2) spaces using α-open sets. Robert.A. and Pious Missier.S.,[10] introduced the concept of semi*α-T_i(i=0,1,2) spaces using semi*α-open sets and investigated their properties.

In this paper, we define regular*-T_0, regular*-T_1 and regular*-T_2 spaces using regular*-open sets and investigate their properties. We further study the relationships among themselves and with known axioms T_0, T_1, T_2 and semi-T_0, semi-T_1, semi-T_2.

Preliminaries: Throughout this paper (X,τ) will always denote topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X,τ), *Cl*(A) and *Int*(A) respectively denote the closure and the interior of A in X.

Definition 2.1: A subset A of a topological space (X, τ) is called (i) **generalized closed** (briefly g-closed) if *Cl(A)⊆U* whenever A⊆U and U is open.
(ii) **generalized open** (briefly g-open) if X\A is g-closed in X.

Definition 2.2: Let A be a subset of X. Then
(i) **generalized closure** of A is defined as the intersection of all g-closed sets containing A and is denoted by *Cl*(A)*.
(ii) **generalized interior** of A is defined as the union of all g-open subsets of A and is denoted by *Int*(A)*.

Definition 2.3: A subset A of a topological space (X,τ) is (i) **Regular*-open** (resp. pre-open, regular-open, semi-open) if *A=Int(Cl*(A)) (resp. A⊆Int(Cl(A)), A=Int(Cl(A)), A⊆Cl(Int(A)))*.
(ii) **Reguler*-closed** (resp. pre-closed, regular-closed, semi-closed) if *A=Cl(Int*(A)) (resp. Cl(Int(A))⊆A, A=Cl(Int(A)), Int(Cl(A))⊆A)*.
The class of all regular*-open (resp. regular*-closed) sets is denoted by R*O(X,τ) (resp. R*C(X,τ)).

Definition 2.4: Let A be a subset of X. Then
(i) the **regular*-closure** of A is defined as the intersection of all regular*-closed sets containing A and is denoted by *r*Cl(A)*.
(ii) the **regular*-interior** of A is defined as the union of all regular*-open sets of X contained and is denoted by *r*Int(A)*.

Theorem 2.5: (i) Every regular*-open set is open. (ii). Every regular*-open set is pre-open

Definition 2.6: A function $f:X\to Y$ is said to be

(i) **regular*-closed** if $f(V)$ is regular*-closed in Y for every closed set V in X.
(ii) **regular*-continuous** if $f^{-1}(V)$ is regular*-open in X for every open set V in Y.
(iii) **regular*-irresolute** if $f^{-1}(V)$ is regular*-open in X for every regular*-open set V in Y.
(iv) **pre-regular*-closed** if $f(V)$ is regular*-closed in Y for every regular*-closed set V in X.
(v) **regular*-open** if $f(V)$ is regular*-open in Y for every open set V in X.
(vi) **pre-regular*-open** if $f(V)$ is regular*-open in Y for every regular*-open set V in X.

Theorem 2.7: Let $A\subseteq X$ and let $x\in X$ and $r^*Cl(A)$ be regular*-closed. Then $x\in r^*Cl(A)$ if and only if every regular*-open set in X containing x intersects A.
Definition 2.8: A topological space X is said to be $\mathbf{T_0}$ **(semi-$\mathbf{T_0}$, α-$\mathbf{T_0}$, pre-$\mathbf{T_0}$)** if whenever x and y are distinct points in X, there is an open (semi-open, α-open, pre-open) set in X containing one of x and y but not the other.
Definition 2.9: A topological space X is said to be $\mathbf{T_1}$ **(semi-$\mathbf{T_1}$, α-$\mathbf{T_1}$, pre-$\mathbf{T_1}$)** if whenever x and y are distinct points in X, there are open (semi-open, α-open, pre-open) sets U and V in X such that U containing x but not y and V containing y but not x.
Definition 2.10: A topological space X is said to be $\mathbf{T_2}$ **(semi-$\mathbf{T_2}$, α-$\mathbf{T_2}$, pre-$\mathbf{T_2}$)** if whenever x and y are distinct points in X, there are open (semi-open, α-open, pre-open) sets U and V in X containing x and y respectively
Lemma 2.11: A topological space X is T_1 if and only if $\{x\}$ is closed in X for every $x\in X$.

Regular*-$\mathbf{T_0}$ Spaces:
Definition 3.1.1: A space X is said to be **regular*-$\mathbf{T_0}$** if whenever x and y are distinct points in X there is a regular*-open set in X containing one of x and y but not the other.
Theorem 3.1.2: (i) Every regular*-T_0 space is T_0.
(ii) Every regular*-T_0 space is pre-T_0.
(iii) Every regular*-T_0 space is semi-T_0.
(iv) Every regular*-T_0 space is α-T_0.
Proof: (i) Suppose X is a regular*-T_0 space. Let x and y be two distinct points in X. Since X is regular*-T_0, there exists a regular*-open set U containing one of x and y but not the other. By Theorem 2.5, U is an open set. Hence X is T_0.
(ii) Follows from Definition 3.1.1 and Theorem 2.5
(iii) Follows from (i) and the fact that every T_0 space is semi-T_0.
(iv) Follows from (i) and the fact that every T_0 space is α-T_0.

Theorem 3.1.3: If a space X is regular*-T_0, then the regular*-closure of distinct points are distinct.
Proof: Let x and y be two distinct points of a regular*-T_0 space X. Then by definition, there exists a regular*-open set U containing one of x and y but not the other. If $x\in U$ and $y\notin U$, then U is a regular*-open set containing x that does not intersect $\{y\}$. By Theorem 2.7, $x\notin r^*Cl(\{y\})$. But $x\in r^*Cl(\{x\})$, so we get $r^*Cl(\{x\})\neq r^*Cl(\{y\})$. Similarly we can prove the case when $y\in U$ and $x\notin U$. Thus the regular*-closure of x and y are distinct.

Theorem 3.1.4: Let $f: X\to Y$ be a bijection. Then the following are true:

(i) If f is regular*-open and X is T_0, then Y is regular*-T_0.
(ii) If f is pre-regular*-open and X is regular*-T_0, then Y is regular*-T_0.
(iii) If f is regular*-continuous and Y is T_0, then X is regular*-T_0.
(iv) If f is regular*-irresolute and Y is regular*-T_0, then X is regular*-T_0.

Proof: (i) Suppose f is regular*-open and X is T_0. Let $y_1 \neq y_2 \in Y$. Since f is a bijection, there exist x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since X is T_0, there exists an open set U in X containing one

of x_1 and x_2 but not the other. Since *f* is regular*-open *f*(U) is a regular*-open set in Y containing y_1 or y_2 but not the other. Thus Y is regular*-T_o.
(ii) Let *f* be pre-regular*-open and X be regular*-T_o. Let $y_1 \neq y_2 \in Y$. Since *f* is a bijection, there exist x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since X is regular*-T_o, there exists a regular*-open set U in X containing one of x_1 and x_2 but not the other. Since *f* is pre-regular*-open *f*(U) is a regular*-open set in Y containing y_1 or y_2 but not the other. Thus Y is regular*-T_o.
(iii) Suppose *f* is regular*-continuous and Y is T_o. Let $x_1 \neq x_2 \in X$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since *f* is a bijection, $y_1 \neq y_2$. Since Y is T_o, there exists an open set V in Y containing one of y_1 and y_2 but not the other. Since *f* is regular*-continuous, $f^{-1}(V)$ is a regular*-open set in X containing one of x_1 and x_2 but not the other. Thus X is regular*-T_o.
(iv). Let *f* be regular*-irresolute and Y be regular*-T_o. Let $x_1 \neq x_2 \in X$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since *f* is a bijection, $y_1 \neq y_2$. Since Y is regular*-T_o, there exists an open set V in Y containing one of y_1 and y_2 but not the other. Since *f* is regular*-irresolute, $f^{-1}(V)$ is a regular*-open set in X containing one of x_1 and x_2 but not the other. Thus X is regular*-T_o.

Regular*-T_1 Spaces:
Definition 3.2.1: A space X is said to be **regular*-T_1** if whenever x and y are distinct points in X, there are regular*-open sets in X containing each but not the other.

Theorem 3.2.2: (i) Every regular*-T_1 space is T_1.
(ii) Every regular*-T_1 space is pre-T_1.
(iii) Every regular*-T_1 space is α-T_1.
(iv) Every regular*-T_1 space is semi-T_1.
(v) Every regular*-T_1 space is regular*-T_o.
Proof: (i) Suppose X is a regular*-T_1 space. Let x and y be two distinct points in X. Since X is regular*-T_1, there exist regular*-open sets U and V with $x \in U$ but $y \notin U$ and $y \in V$ and $x \notin V$. By Theorem 2.5, U and V are open sets. Hence X is T_1.
(ii) Follows from Definition 3.2.1 and Theorem 2.5.
(iii) Follows from (i) and the fact that every T_1 space is semi-T_1.
(iv) Follows from (i) and the fact that every T_1 space is α-T_1.
(v). Follows from Definitions.

Theorem 3.2.3: Suppose R*O(X, τ) is closed under arbitrary union, for a topological space X, the following are equivalent:
(i) X is regular*-T_1 space.
(ii) Each one point set in X is regular*-closed in X.
(iii) Each subset of X is the intersection of regular*-open sets containing it.
(iv) The intersection of all regular*-open sets in X containing the point x equals {x}.
Proof: (i)⇒(ii): Suppose X is regular*-T_1. Let $x \in X$, then for every $y \neq x$, there exists a regular*-open set U_y in X containing y but not x. Hence $y \in U_y \subseteq X\backslash\{x\}$. Therefore $X\backslash\{x\} = \cup\{U_y : y \in X\backslash\{x\}\}$. By assumption, $X\backslash\{x\}$ is regular*-open in X. Thus {x} is regular*-closed.
(ii)⇒(iii): Let $A \subseteq X$. Then for each $x \in X\backslash A$, {x} is regular*-closed in X and hence $X\backslash\{x\}$ is regular*-open. Clearly $A \subseteq X\backslash\{x\}$ for each $x \in X\backslash A$. Therefore $A \subseteq \cap\{X\backslash\{x\} : x \in X\backslash A\}$. On the other hand, if $y \notin A$, then $y \in X\backslash A$ and $y \notin X\backslash\{y\}$. This implies $y \notin \cap\{X\backslash\{x\}: x \in X\backslash A\}$. Hence $\cap\{X\backslash\{x\}: x \in X\backslash A\} \subseteq A$. Therefore $A = \cap\{X\backslash\{x\}: x \in X\backslash A\}$ which proves (iii).
(iii)⇒(iv): Taking A = {x}, by (iii) $A = \{x\} = \cap\{U: U$ is regular*-open and $x \in U\}$. This proves (iv).
(iv)⇒(i): Let $x, y \in X$ with $y \neq x$, then $y \notin \{x\} = \cap\{U: U$ is regular*-open and $x \in U\}$. Hence there exists a regular*-open set U containing x but not y. Similarly, there exists a regular*-open set V containing y but not x. Thus X is regular*-T_1.

Theorem 3.2.4: Let *f: X→Y* be a bijection.
(i) If *f* is regular*-continuous and Y is T_1, then X is regular*-T_1.
(ii) If *f* is regular*-irresolute and Y is regular*-T_1, then X is regular*-T_1.

(iii) If f is regular*-open and X is T_1, then Y is regular*-T_1.
(iv) If f is pre-regular*-open and X is regular*-T_1, then Y is regular*-T_1.
(v) If f is regular*-closed and X is T_1, then Y is regular*-T_1.
(vi) If f is pre-regular*-closed and X is regular*-T_1, then Y is regular*-T_1.

Proof: (i) Suppose f is regular*-continuous bijection and Y is T_1. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is one to one, $y_1 \neq y_2$. Since Y is T_1, there exist open sets U and V such that $y_1 \in U$ but $y_2 \notin U$ and $y_2 \in V$ but $y_1 \notin V$. Since f is a bijection, $x_1 \in f^{-1}(U)$ but $x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V)$ but $x_1 \notin f^{-1}(V)$. Since f is regular*-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are regular*-open sets in X. Thus X is regular*-T_1.

(ii) Suppose f is regular*-irresolute bijection and Y is regular*-T_1. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is one to one, $y_1 \neq y_2$. Since Y is regular*-T_1, there exist regular*-open sets U and V such that $y_1 \in U$ but $y_2 \notin U$ and $y_2 \in V$ but $y_1 \notin V$. Since f is a bijection, $x_1 \in f^{-1}(U)$ but $x_2 \notin f^{-1}(U)$ and $x_2 \in f^{-1}(V)$ but $x_1 \notin f^{-1}(V)$. Since f is regular*-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are regular*-open sets in X. Thus X is regular*-T_1.

(iii) Suppose f is regular*-open bijection and X is T_1. Let $y_1 \neq y_2 \in Y$. Since f is a bijection, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since X is T_1, there exist open sets U and V in X such that $x_1 \in U$ but $x_2 \notin U$ and $x_2 \in V$ but $x_1 \notin V$. Since f is regular*-open, $f(U)$ and $f(V)$ are regular*-open sets in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Since f is a bijection, $y_2 = f(x_2) \notin f(U)$ and $y_1 = f(x_1) \notin f(V)$. Thus Y is regular*-T_1.

(iv) Suppose f is a pre-regular*-open bijection and X is regular*-T_1. Let $y_1 \neq y_2 \in Y$. Since f is a bijection, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since X is regular*-T_1, there exist regular*-open sets U and V in X such that $x_1 \in U$ but $x_2 \notin U$ and $x_2 \in V$ but $x_1 \notin V$. Since f is pre-regular*-open, $f(U)$ and $f(V)$ are regular*-open sets in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Since f is a bijection, $y_2 = f(x_2) \notin f(U)$ and $y_1 = f(x_1) \notin f(V)$. Thus Y is regular*-T_1.

(v) Suppose f is a regular*-closed bijection and X is T_1. Let $y \in Y$. Since f is bijection, there exists $x \in X$ such that $f(x) = y$. Since X is T_1, by Lemma 2.11, $\{x\}$ is closed in X. Since f is a regular*-closed map, $f(\{x\}) = \{y\}$ is regular*-closed. Since every singleton set in Y is regular*-closed, by Theorem 3.2.3, Y is regular*-T_1.

(vi) Suppose f is a pre-regular*-closed bijection and X is regular*-T_1. Let $y \in Y$. Since f is bijection, there exists $x \in X$ such that $f(x) = y$. Since X is regular*-T_1, by Theorem 3.2.3, $\{x\}$ is regular*-closed in X. Since f is a pre-regular*-closed map, $f(\{x\}) = \{y\}$ is regular*-closed. Since every singleton set in Y is regular*-closed, by Theorem 3.2.3, Y is regular*-T_1.

Regular*-T_2 Space:

Definition 3.3.1: A space X is said to be **regular*-T_2** if whenever x and y are distinct points in X, there are disjoint regular*-open sets U and V in X containing x and y respectively.

Theorem 3.3.2: (i) Every regular*-T_2 space is T_2.
(ii) Every regular*-T_2 space is pre-T_2.
(iii) Every regular*-T_2 space is semi-T_2.
(iv). Every regular*-T_2 space is α-T_2.
(v).Every regular*-T_2 space is regular*-T_1.
(vi). Every regular*-T_2 space is regular*-T_0.

Proof: (i) Suppose X is a regular*-T_2 space. Let x and y be two distinct points in X. Since X is regular*-T_2, there exist disjoint regular*-open sets U and V such that $x \in U$ and $y \in V$. By Theorem 2.5, U and V are open sets such that $x \in U$ and $y \in V$. Hence X is T_2.
(ii) Follows from Definition 3.3.1 and Theorem 2.5.
(iii) Follows from the fact that every T_2 space is semi-T_2.
(iv). Follows from the fact that every T_2 space is α-T_2.
(v). Follows from Definitions.
(vi). Follows from Definitions and Theorem 3.2.2.

Theorem 3.3.3: For a topological space X the following are equivalent:
(i) X is regular*-T_2 space.
(ii) Let $x \in X$, then for each $y \neq x$, there exists a regular*-open set U such that $x \in U$ and $y \notin r^*Cl(U)$.

(iii) For each $x \in X$, $\cap\{r^*Cl(U): U \in R^*O(X) \text{ and } x \in U\} = \{x\}$.

Proof: (i)⇒(ii): Suppose X is a regular*-T_2 space. Let $x \in X$ and $y \in X$ with $y \neq x$, then there exist disjoint regular*-open sets U and V such that $x \in U$ and $y \in V$. Since V is regular*-open, X\V is regular*-closed and U⊆X\V. This implies that $r^*Cl(U) \subseteq X\backslash V$. Since $y \notin X\backslash V$, $y \notin r^*Cl(U)$.

(ii)⇒(iii): If $y \neq x$ then there exists a regular*-open set U such that $x \in U$ and $y \notin r^*Cl(U)$. Hence $y \notin \cap\{r^*Cl(U): U \in R^*O(X) \text{ and } x \in U\}$. This proves (iii).

(iii)⇒(i): Let $y \neq x$ in X. Then $y \notin \cap\{r^*Cl(U): U \in R^*O(X) \text{ and } x \in U\}$. This implies that there exists a regular*-open set U such that $x \in U$ and $y \notin r^*Cl(U)$. Take $V = X\backslash r^*Cl(U)$ is a regular*-open set, then $y \in V$. Now $U\cap V = U\cap(X\backslash r^*Cl(U)) \subseteq U\cap(X\backslash U) = \phi$. This proves (i).

Theorem 3.3.4: Let $f: X\to Y$ be a bijection.

(i) If f is regular*-continuous and Y is T_2, then X is regular*-T_2.

(ii) If f is regular*-irresolute and Y is regular*-T_2, then X is regular*-T_2.

(iii) If f is regular*-open and X is T_2, then Y is regular*-T_2.

Proof: (i) Suppose f is regular*-continuous and Y is T_2. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is one to one $y_1 \neq y_2$. Since Y is T_2, there exist disjoint open sets U and V containing y_1 and y_2 respectively. Since f is regular*-continuous bijection, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint regular*-open sets in X containing x_1 and x_2 respectively. Thus X is regular*-T_2.

(ii) Proof is similar to (i).

(iii). Suppose f is regular*-open and X is T_2. Let $y_1 \neq y_2 \in Y$. Since f is a bijection, there exist x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since X is T_2, there exist disjoint open sets U and V in X such that $x_1 \in U$ and $x_2 \in V$. Since f is regular*-open in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Since f is bijection $f(U)$ and $f(V)$ are disjoint in Y. Thus Y is regular*-T_2.

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