THE CONNECTED TOTAL EDGE MONOPHONIC NUMBER OF GRAPH

P. Arul Paul Sudhahar
Department of Mathematics,
Rani Anna Govt. College (W), Tirunelveli – 627 008, Tamilnadu, India
arulpaulsudhar@gmail.com

V.K. KalaiVani
Research Scholar (Reg.No:12445),
Rani Anna Govt. College (W), Tirunelveli – 627008, Tamilnadu, India.
Manonmaniam Sundaranar University, Tirunelveli-627012, Tamilnadu, India
kalaivanisaji@gmail.com

Abstract: A set M of vertices of a connected graph G is a monophonic set if every vertex of G lies on an x-y monophonic path for some elements x and y in M. The minimum cardinality of a monophonic set of G is the monophonic number of G, denoted by $m(G)$. A total monophonic set of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality of a total monophonic set of G is the total monophonic number denoted by $m_t(G)$. A total edge monophonic set of a graph G is an edge monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total edge monophonic set of G is the total edge monophonic number of G, and is denoted by $em_t(G)$. A connected total edge monophonic set of a graph G is a total edge monophonic set M such that the subgraph $\langle M\rangle$ induced by M is connected . The minimum cardinality of a connected total edge monophonic set of G is the connected total edge monophonic number of G and is denoted by $em_{ct}(G)$. The connected total edge monophonic number of some connected graphs are realized. It is proved that if p, a, b are positive integers such that $4 \leq a \leq b \leq p$, then there exists a connected graph G of order p, $em_t(G) = a$ and $em_{ct}(G) = b$. For any positive integer p, q, r with $p \leq q \leq r$, there exist a connected graph G such that $m(G) = p$, $m_e(G) = q$ and $em_{ct}(G) = r$.

Keywords: Connected Total Monophonic Number, Connected Total Edge Monophonic Number, Monophonic Set, Monophonic Number, Total Edge Monophonic Number.

AMS Subject Classification: 05C12.

1. **Introduction:** By a graph $G = (V, E)$ we mean a simple graph of order at least two. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology, we refer to Harary [3]. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if $\Delta(\langle N(v)\rangle) = |N(v)| - 1$. In particular, every extreme vertex is a semi - extreme vertex and a semi - extreme vertex need not be an extreme vertex.

For any two vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an x-y geodesic. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [4]. An edge geodetic set of G is a subset $S \subset V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets. The connected edge geodetic set of a connected graph is studied in [10].

A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge u_iu_j with $j \geq i + 2$. A u-v path P is called a monophonic path if it is a chordless path. A set M of vertices is a monophonic set if every vertex of G lies on a monophonic path joining some pair of vertices in M, and the minimum cardinality of a monophonic set is the monophonic number $m(G)$. The monophonic number of a graph G was studied in [9]. A set M of vertices of a graph G is an edge monophonic set if every edge of G lies on an $x - y$ monophonic path for some elements x and y in M. The minimum cardinality of an edge monophonic set of G is the edge monophonic number of G, denoted by $em(G)$. The edge monophonic number of a graph was introduced and studied in [8].

A total edge monophonic set of a graph G is an edge monophonic set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality of a total monophonic set of G is the total monophonic number denoted by $m_t(G)$. The Total edge monophonic number of a graph was introduced and studied in [1]. A connected edge monophonic set of a graph G is an edge monophonic set M such that the subgraph $\langle M \rangle$ induced by M is connected. The minimum cardinality of a connected edge monophonic set of G is the connected edge monophonic number of G and is denoted by $em_c(G)$. The connected edge monophonic number of a graph was studied in [7]. A connected total monophonic set of a graph G is a total monophonic set M such that the subgraph $\langle M \rangle$ induced by M is connected. The minimum cardinality of a connected total monophonic set of G is the connected total monophonic number of G and is denoted by $m_{ct}(G)$.The connected total monophonic number of a graph was studied in [2].

A connected total edge monophonic set of a graph G is a total edge monophonic set M such that the subgraph $\langle M \rangle$ induced by M is connected. The minimum cardinality of a connected total edge monophonic set of G is the connected total edge monophonic number of G and is denoted by $em_{ct}(G)$.
The following Theorems are used in the sequel.

Theorem 1.1 [2]: For any non trivial tree T of order p, $m_{ct}(T) = p$.

Theorem 1.2 [1]: Each extreme vertex of G belongs to every total edge monophonic set of G.

Theorem 1.3 [9]: The monophonic number of a tree T is the number of end vertices in G.

Theorem 1.4 [2]:
1. For the complete graph K_n of order $n \geq 2$, $em_t(G) = n$.
2. For any non-trivial tree T of order n with k end vertices, $em_t(G) = k$.
3. For any Wheel Wn ($n \geq 5$) of order n, $em_t(G) = n-1$.

Theorem 1.5 [10]: Every semi-extreme vertex and every cut-vertex of a connected graph G belong to each connected edge geodetic set of G.

Theorem 1.6 [1]: Every semi-extreme vertex of a connected graph G belongs to each total edge monophonic set of G. In particular, if the set M of all semi-extreme vertices of G is an total edge monophonic set of G, then M is the unique minimum total edge monophonic set of G.

Theorem 1.7 [8]: For the complete graph K_n of order $n \geq 2$, $em(G) = n$.

Theorem 1.8[2]: Each extreme vertex and cut vertex of a connected graph G belongs to every connected total monophonic set of G.

2. The Connected Total Edge Monophonic Number Of a Graph:

Definition 2.1: Let G be a connected graph with at least two vertices. A connected total edge monophonic set of G is a total edge monophonic set M such that the subgraph induced by M is connected. The minimum cardinality of a connected total edge monophonic set of G is the connected total edge monophonic number of G and is denoted by $em_{ct}(G)$. A connected total edge monophonic set of cardinality $em_{ct}(G)$ is called an em_{ct} – set of G.

Example 2.2: For the graph G given in Figure 2.1, it is clear that $M_1 = \{v_1, v_2, v_3, v_6, v_8\}$ is an edge monophonic set of G, so that $em(G) = 5$. It is easily seen that no 5-element subset of vertices is a total edge monophonic set. It is clear that $M_2 = \{v_1, v_2, v_3, v_5, v_6, v_8\}$ is a total edge monophonic set of G, so that $em_t(G) = 6$. Here the induced subgraph $\langle M_2 \rangle$ is not connected, so that M_2 is not a connected total edge monophonic set of G. Now it is clear that $M_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8\}$ is a minimum connected total edge monophonic set of G, we have $em_{ct}(G) = 7$.

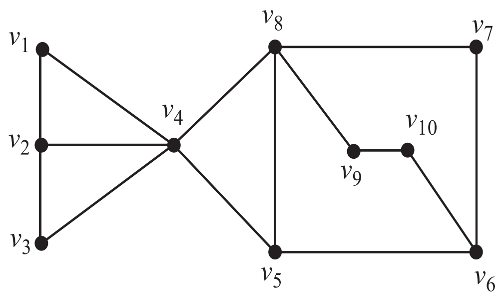

Figure 2.1

Theorem 2.3: For any connected graph G of order p, $2 \leq em_t(G) \leq em_{ct}(G) \leq p$.

Proof: An total edge monophonic set needs at least two vertices and so $em_t(G) \geq 2$. Since every connected total edge monophonic set is also an edge monophonic set, it follows that $em_t(G) \leq em_{ct}(G)$. Also, since the set of all vertices of G is a connected total edge monophonic set of G, $em_{ct}(G) \leq p$. We observe that for the complete graph K_2, $em_{ct}(K_2) = em_t(K_2) = 2$ and for the complete graph K_n ($n \geq 3$), $em_{ct}(G) = em_t(G) = n$. Also all the inequalities in Theorem 2.3 are strict. For the graph G given in Figure 2.1, $em_t(G) = 6$, $em_{ct}(G) = 7$ and $n = 10$.

Theorem 2.4: Every semi-extreme vertex of a connected graph G belongs to each connected total edge monophonic set of G. In particular, if the set M of all semi-extreme vertices of G is a connected total edge monophonic set of G, then M is the unique minimum connected total edge monophonic set of G.

Proof: Let M be a connected total edge monophonic set of G. Let v be a semi-extreme vertex of G. Suppose that $v \notin M$. Let u be a vertex of $\langle N(v) \rangle$ such that $deg_{\langle N(v) \rangle}(u) = |N(v)| - 1$. Let $u_1, u_2, \ldots, u_k$ ($k \geq 2$) be the neighbors of u in $\langle N(v) \rangle$. Since M is a connected total edge monophonic set of G, the edge uv lies on the monophonic path $P : x, x_1, \ldots, u_i, u, v, u_j, \ldots, y$, where $x,y \in M$. Since v is a semi-extreme vertex of G, u and u_j are adjacent in G and so P is not a monophonic path of G, which is a contradiction. Therefore M is the unique minimum connected total edge monophonic set of G. Hence, every semi-extreme vertex of a connected graph G belongs to each connected total edge monophonic set of G. Since every connected total edge monophonic set is also total edge monophonic set, the result follows from Theorem 1.6.

Corollary 2.5: For any connected graph G of order n with k semi-extreme vertices, $\max\{2,k\} \leq em_{ct}(G) \leq n$.

Proof: This follows from Theorem 2.3 and 2.4.

Corollary 2.6: Each extreme vertex of a graph G belongs to every connected total edge monophonic set of G.

Proof: Since every extreme vertex of G is a semi-extreme vertex of G, the result follows from Theorem 2.4.

Corollary 2.7: For the complete graph K_n ($n \geq 2$), $em_{ct}(K_n) = n$.

The converse of Corollary 2.7 need not be true. For the graph G given in Figure 2.2, each vertex is a semi-extreme and $M = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is an em_{ct}-set of G. Therefore, $em_{ct}(G) = 6$ and G is not a complete graph.

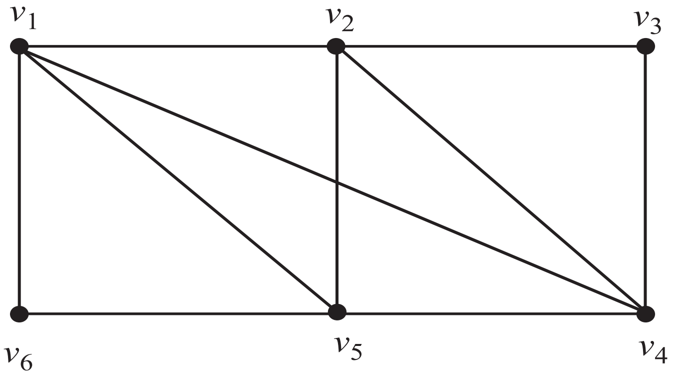

Figure 2.2

Theorem 2.8: Let G be a connected graph with cut-vertices and M be a connected total edge monophonic set of G. If v is a cut-vertex of G, then every component of G-v contains an element of M.

Proof: Let v be a cut vertex of G and M be a connected total edge monophonic set of G. Suppose there exists a component, say G_1 of $G - v$ such that G_1 contains no vertex of M. By Corollary 2.6, M contains all the extreme vertices of G and hence it follows that G_1 does not contains any extreme vertex of G. Thus G_1 contains at least one edge, say xy. Since M is a connected total edge monophonic set, xy lies on the u-w monophonic path $P : u, u_1, u_2, \ldots, v, \ldots, x, y, \ldots, v_1, \ldots, v, \ldots, w$. Since v is a cut vertex of G, the u-x and y-w sub paths of P both contain v and so P is not a monophonic path, which is a contradiction. Therefore, every component of G-v contains an element of M.

Theorem 2.9: Every cut-vertex of a connected graph G belongs to every connected total edge monophonic set of G.

Proof: Let G be a connected graph and M be a connected total edge monophonic set of G. Let v be a cut-vertex of G and $G_1, G_2, \ldots, G_r$ $(r \geq 2)$ the components of G-v. By Theorem 2.8, M contains at least one vertex from each G_i $(1 \leq i \leq r)$. Since the subgraph induced by M is connected, it follows that $v \in M$.

Corollary 2.10: For any connected graph G of order n with k semi-extreme vertices and l cut-vertices, $\max\{2, k+l\} \leq em_{ct}(G) \leq n$.

Proof: This follows from Theorems 2.3, 2.4 and 2.9.

Corollary 2.11:

1. For any tree T of order n, $em_{ct}(T) = n$.
2. Let G be any connected graph of order $n \geq 3$.Then $em_t(G) = em_{ct}(G) = 3$ if and only if $G = K_3$

Theorem 2.12: For the complete bipartite graph $G = K_{m,n}$ $(2 \leq m \leq n)$,

1. $em_{ct}(G) = 2$ *if* $m = n = 1$
2. $em_{ct}(G) = n + 1$ *if* $m = 1$, $n \geq 2$.
3. $em_{ct}(G) = \min\{m,n\}+1$ *if* $m, n \geq 2$.

Proof: Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the partite set of G. (i) This follows from corollary 2.7. (ii) This follows from Corollary 2.11(i) (iii) Let $m, n \geq 2$. First assume that $m < n$. Let $M = U \cup \{w_1\}$.We prove that M is a minimum connected total edge monophonic set of G. Note that any u-v monophonic path in G is of length at most 2. Every edge u_iw_j $(1 \leq i \leq m, 1 \leq j \leq n)$ lies on the monophonic path u_i, w_j, u_k for any $k \neq i$ and so M is a connected total edge monophonic set of G. Let T be any set of vertices such that $|T| < |M|$. If $T \subseteq U$ or $T \subseteq W$, then T cannot be a connected total edge monophonic set of G. If T is such that T contains vertices from U and W such that $u_i \notin T$ and $w_i \notin$ T. Then clearly the edge u_iw_j does not lie on a monophonic path joining two vertices of T, so that T is not a connected total edge monophonic set of G. Thus in any case T is not a connected total edge monophonic set of G. Hence M is a minimum connected total edge monophonic set, so that $em_{ct}(G) = \min\{m, n\}+1$.

Theorem 2.13: For any cycle C_n ($n \geq 3$), $em_{ct}(C_n) = 3$.
Proof: Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order $n \geq 3$. It is clear that no 2-element subset of vertices is a connected total edge monophonic set of G. Now $M = \{u_1, u_2, u_3\}$ is a connected total edge monophonic set of G, so that $em_{ct}(G) = 3$.

Theorem 2.14: For any connected graph G of order n, $em_{ct}(G) = 2$ if and only if $G = K_2$.
Proof: If $G = K_2$, then $em_{ct}(G) = 2$. Conversely, let $em_{ct}(G) = 2$. Let $M = \{u, v\}$ be a minimum connected total edge monophonic set of G. Then uv is an edge. If $G \neq K_2$, then there exists an edge xy different from uv, and the edge xy does not lie on any u-v monophonic path, so that M is not a connected total edge monophonic set, which is a contradiction. Thus $G = K_2$.

Theorem 2.15: Let G be a connected graph of order n. Then $em_{ct}(G) = n$ if and only if every vertex of G is either a cut-vertex or a semi-extreme vertex.
Proof: Let $em_{ct}(G) = n$. Then $M = V$ is the only connected edge monophonic set of G. Suppose that there exists a vertex u such that u is neither a semi-extreme vertex nor a cut-vertex of G. Since u is not a semi-extreme vertex, for each $v \in N(u)$, there exists a vertex $w \in N(u)$ such that $w \neq v$ and vw is not an edge of G. Now, we show that $M = V - \{u\}$ is an edge total monophonic set of G. Let vu be any edge of G. Then vu lies on the monophonic path $P : v, u, w$. Also, any xy not incident with u lies on the x-y monophonic path itself. Hence M is an edge total monophonic set of G. Since u is not a cut-vertex of G, the subgraph induced by M is connected. Therefore, M is a connected total edge monophonic set of G, so that $em_{ct}(G) \leq n-1$, which is a contradiction. Hence every vertex of G is either a semi-extreme vertex or a cut-vertex. The converse follows from Theorems 2.4 and 2.9.

3. Realization Results:
Theorem 3.1: If n, a, b are positive integers such that $4 \leq a \leq b \leq n$, then there exists a connected graph G of order n, $em_t(G) = a$ and $em_{ct}(G) = b$.
Proof: We prove this theorem by considering four cases.

Case 1: $a = b = n$. Let $G = K_n$. Then $em_t(G) = em_{ct}(G) = n$.
Case 2: $a < b < n$. Let $P_{b-a+3} : u_1, u_2, u_3, \ldots, u_{b-a+3}$ be a path of order $b-a+3$. Add $n-b+a-3$ new vertices $v_1, v_2, \ldots, v_{n-b}$ and $w_1, w_2, \ldots, w_{a-3}$ to P_{b-a+3} and join $v_1, v_2, \ldots, v_{n-b}$ with both u_1 and u_3 and also join each of w_i ($1 \leq i \leq a-3$) with u_{b-a+3}. Let G be the graph in Figure3.4 of order n. Let $M = \{w_1, w_2, \ldots, w_{a-3}, u_{b-a+3}\}$ be the extreme vertices of G. By Theorem 1.2, every total monophonic set of G contains M. Since $M \cup \{u_1, u_2\}$ is a total edge monophonic set of G and it follows from Theorem 1.2, $em_t(G) = a$. Let $M_1 = \{w_1, w_2, \ldots, w_{a-3}, u_3, u_4, \ldots, u_{b-a+3}\}$ be the set of all extreme vertices and cut-vertices of G. By Theorem 2.4 and 2.9, every connected total edge monophonic set of G contains M_1. Since $M_1 \cup \{u_1, u_2\}$ is connected total edge monophonic set of G, so that $em_{ct}(G) = b$.

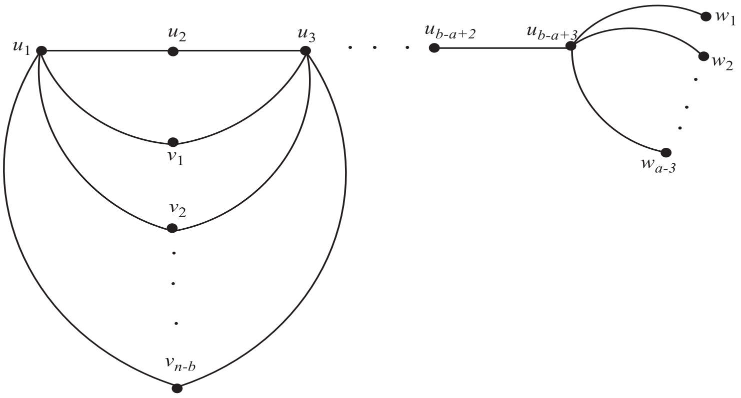

Figure 3.1

Case 3: $a = b < n$. Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Add new vertices $v_1, v_2, \ldots, v_{n-a}$ and join each v_i ($1 \le i \le n-a$) with u_1 and u_3. Also, add new vertices $w_1, w_2, w_3, \ldots, w_{a-3}$ and join each w_i ($1 \le i \le a-3$) with u_1, thereby obtaining graph G in Figure 3.5 of order n. Let $M = \{w_1, w_2, \ldots, w_{a-3}, u_3\}$. By Theorem 1.7, every total edge monophonic set as well as every connected total monophonic set of G contains M. Since $M \cup \{u_2, u_3\}$ is a connected total edge monophonic set of G, so that $em_t(G) = em_{ct}(G) = b$.

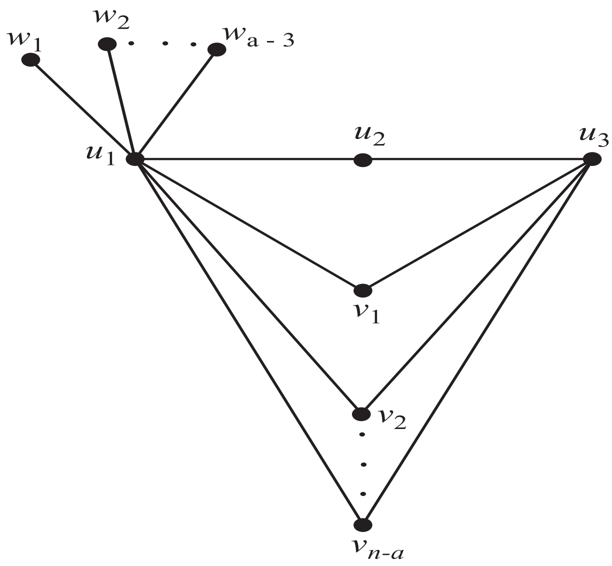

Figure 3.2

Case 4. $a < b = n$. Let $P_{b-a+3} : u_1, u_2, u_3, \ldots, u_{b-a+3}$ be a path of order $b-a+3$. Add $a-3$ new vertices $v_1, v_2, \ldots, v_{a-3}$ to P_{b-a+3} and join each v_i ($1 \le i \le a-3$) with u_{b-a+3}, thereby obtaining the graph G in Figure 3.6 of order $n = b$. Let $M = \{u_1, u_2, v_1, v_2, \ldots, v_{a-3}, u_{b-a+3}\}$. Since M is a total edge monophonic set of G, so that $em_t(G) = a$. Let$M_1 = \{u_2, u_3, \ldots, u_{b-a+3}\}$ be the set of all cut-vertices of G. By Theorem 2.9, every connected total edge monophonic set of G contains M_1. It is easily verified that the set $M \cup M_1$ is the unique connected total edge monophonic set of G, so that $em_{ct}(G) = b = n$.

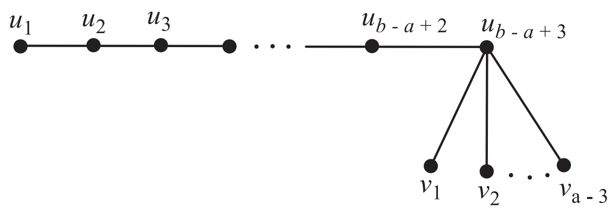

Figure 3.3

Theorem 3.2: For any positive integers p, q, r with $p \le q \le r$, there exist a connected graph G such that $m(G) = p$, $m_e(G) = q$ and $em_{ct}(G) = r$.

Proof: Let G be the graph having the path P of order $u_1, u_2, u_3, \ldots, u_{r-q+2}$ and by adding q-2 vertices $v_1, v_2, \ldots, v_{q-p}$ and $w_1, w_2, \ldots, w_{p-2}$ and join each v_i with u_1 *and* u_3 and join each w_i *with* u_2.

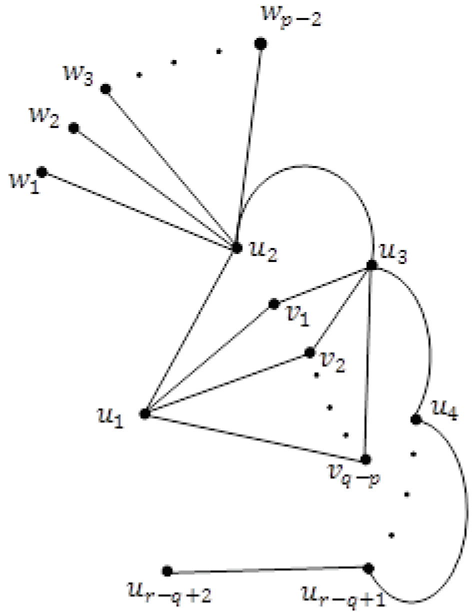

Figure 3.4

Then $M = \{ w_1, w_2, \ . \ . \ . \ , w_{p-2}, u_1, u_{r-q+2}\}$ is a minimum monophonic set of G, so that $m(G) = p$. Now $M_1 = M \cup \{v_1, v_2, \ . \ . \ . \ , v_{q-p}\}$ is a minimum edge monophonic set. Thus $em(G) = q$. Take $M_2 = M_1 \cup \{u_2, u_3, . \ . \ . \ , u_{r-q+2}\}$. Clearly M_2 is a connected total edge monophonic set of G. Hence $em_{ct}(G) = |M_2| = q + (r - q + 1) - 1 = r$.

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