qpI- CONNECTEDNESS IN IDEAL BITOPOLOGICAL SPACES

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Abstract: The purpose of this paper is to introduce and study the notion of qpI- connectedness in ideal bitopological spaces. We shall also study the notions of qpI- separated sets in ideal bitopological spaces

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1. Introduction and Preliminaries

In 1961 Kelly introduced the concept of bitopological spaces as an extension of topological spaces [6]. A bitopological space (X, τ_1, τ_2) is a nonempty set X equipped with two topologies τ_1 and τ_2 [6]. The study of quasi open sets in bitopological spaces was initiated by Datta in 1971 [1]. In a bitopological space (X, τ_1, τ_2) a set A of X is said to be quasi open if it is a union of a τ_1-open set and a τ_2-open set [1]. Complement of a quasi open set is termed quasi closed. Every τ_1-open (resp. τ_2-open) set is quasi open but the converse may not be true. Any union of quasi open sets of X is quasi open in X. The intersection of all quasi closed sets which contains A is called quasi closure of A. It is denoted by qCl(A))[1]. The union of quasi open subsets of A is called quasi interior of A. It is denoted by qInt(A) [1].

Mashhour introduced the concept of preopen sets in topology [12]. A subset A of a topological space (X, τ) is called preopen if $A \subset Int(Cl(A))$ [12]. Further, in 1995 Tapi introduced the concept of quasi preopen sets in bitopological spaces [14]. A set A in a bitopological space (X, τ_1, τ_2) is called quasi preopen if it is a union of a τ_1-preopen set and a τ_2-preopen set [14]. Complement of a quasi preopen set is called quasi pre closed. Every τ_1-preopen (τ_2-preopen, quasi open) set is quasi preopen but the converse may not be true. Any union of quasi preopen sets of X is a quasi preopen set in X. The intersection of all quasi pre closed sets which contains A is called quasi pre closure of A. It is denoted by qpCl(A) [14]. The union of quasi preopen subsets of A is called quasi pre interior of A. It is denoted by qpInt(A) [14].

The study of ideal topological spaces was initiated by Kuratowski [11] and Vaidyanathaswamy [15]. Applications to various fields were further investigated by Dontchev [2], Jankovic and Hamlett [5], Nasef and Mahmoud [13] and others.

An Ideal I on a topological space (X, τ) is a non empty collection of subsets of X which satisfies:

i. $A \in I$ and $B \subset A \Rightarrow B \in I$ and
ii. $A \in I$ and $B \in I \Rightarrow A \cup B \in I$

An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I). If $\mathcal{P}(X)$ is the set of all subsets of X, in a topological space (X, τ) a set operator $(.)^*: \mathcal{P}(X) \to \mathcal{P}(X)$ is called the local mapping [2] of A with respect to τ and I and is defined as follows:

$A^*(\tau, I) = \{x \in X \mid U \cap A \notin I, \forall\ U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$.

Definition 1.1. [9]. If (X, τ_1, τ_2) is a bitopological space then (X, τ_1, τ_2, I) is an ideal bitopological space if I is an ideal on X.

In 2010 Jafari and Rajesh defined quasi local mapping of A with respect to τ_1, τ_2 and I and defined it as follows $A^*_q(\tau_1, \tau_2, I) = \{x \in X \mid U \cap A \notin I, \forall$ quasi open set U containing x$\}$ [4].

Definition1.2. [7]. Given an ideal bitopological space (X, τ_1, τ_2, I) the quasi pre-local mapping of A with respect to τ_1, τ_2 and I denoted by $A^*_{qp}(\tau_1, \tau_2, I)$ (more generally as A^*_{qp}) is defined as $A^*_{qp}(\tau_1, \tau_2, I) = \{x \in X \mid U \cap A \notin I, \forall$ quasi pre-open set U containing x$\}$

Definition1.3. [7]. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is qpI- open if $A \subset$ qpInt(A^*_{qp}) and qpI- closed if its complement is qpI- open.

Definition1.3. [7]. A mapping f: $(X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is called a qpI- continuous if $f^{-1}(V)$ is a qpI- open set in X for every quasi open set V of Y

Definition1.4. [7]. In an ideal bitopological space (X, τ_1, τ_2, I) the quasi $*$-pre closure of A of X denoted by qpCl*(A) is defined by qpCl$^*(A) = A \cup A^*_{qp}$

Definition1.5. [7]. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be a qpI-neighbourhood of a point $x \in X$ if $\exists$ a qpI- open set O such that $x \in O \subset A$

Definition1.6. [7]. Let A be a subset of an ideal bitopological space (X, τ_1, τ_2, I) and $x \in X$. Then x is called a qpI-interior point of A if $\exists$ V a qpI- open set in X such that $x \in V \subset A$. The set of all qpI-interior points of A is called the qpI- interior of A and is denoted by qpIInt(A).

Definition1.7. [7]. Let A be a subset of an ideal bitopological space (X, τ_1, τ_2, I) and $x \in X$. Then x is called a qpI-cluster point of A, if $V \cap A \neq \emptyset$, for every qpI- open set V in X. The set of all qpI-cluster points of A denoted by qpICl(A) is called the qpI-closure of A .

Definition 1.8. [3]. An ideal topological space (X, τ, I) is called *-connected if X cannot be written as the disjoint union of a nonempty open set and a nonempty *-open set.

Definition 1.9. [8]. An ideal bitopological space (X, τ_1, τ_2, I) is called pairwise *-connected if X cannot be written as the disjoint union of a nonempty τ_i open set and a nonempty τ^*_j -open set. $\{i, j = 1, 2;\ i \neq j\}$

Definition 1.10. [8]. Nonempty subsets A, B of an ideal bitopological space (X, τ_1, τ_2, I), are called pairwise *-separated if τ_iCl*(A) $\cap$ B = A $\cap$ τ_jCl(B) = ϕ. $\{i, j = 1, 2;\ i \neq j\}$

2. *qpI- Connectedness in Ideal Bitopological Spaces*

Definition 2.1. An ideal topological space (X, τ_1, τ_2, I) is called qpI- connected if X cannot be written as the disjoint union of a nonempty quasi open set and a nonempty qpI- open set.

Definition 2.2. Nonempty subset A, B of an ideal bitopological space (X, τ_1, τ_2, I) are called qpI-separated if qCl(A) $\cap$ B = A $\cap$ qpICl(B) = ϕ.

Theorem 2.1. If A, B are -separated sets of an ideal bitopological space (X, τ_1, τ_2, I) and $A \cup B \in \tau_1 \cap \tau_2$ then A is qpI- open and B is quasi open.

Proof: Since A and B are qpI-separated in X, then B = (A $\cup$ B) $\cap$ (X - qCl(A)). Since A$\cup$ B is biopen and qCl(A) is quasi closed in X, B is quasi open in X. Similarly A = (A $\cup$ B) $\cap$ (X - qpICl(B)) and we obtain that A is qpI-open in X.

Theorem 2.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space and A, B $\subset$ Y $\subset$ X. Then A and B are qpI-separated in Y if and only if A, B are qpI-separated in X.

Proof: It follows from qCl(A) $\cap$ B = A $\cap$ qpICl(B) = ϕ and the fact that A, B $\subset$ Y $\subset$ X.

Theorem 2.3. If f: $(X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)$ is a qpI- continuous onto mapping. Then if $(X, \sigma_1, \sigma_2, I)$ is a qpI-connected ideal bitopological space (Y, σ_1, σ_2) is also quasi connected.

Proof: It is known that connectedness is preserved by continuous surjections. Hence every qp**I**-open set is also quasi open. Hence, qpI- connected space is also quasi connected.

Definition 2.3. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is called qpI-*s*-connected if A is not the union of two nonempty qpI-separated sets in (X, τ_1, τ_2, I).

Theorem 2.4. Let Y be a biopen subset of an ideal bitopological space (X, τ_1, τ_2, I)
The following are equivalent:

i. Y is qpI-*s*-connected in (X, τ_1, τ_2, I)
ii. Y is qpI- connected in (X, τ_1, τ_2, I).

Proof: i) $\Rightarrow$ ii) Let Y be qpI-*s*-connected in (X, τ_1, τ_2, I) and suppose that Y is not qpI-connected in (X, τ_1, τ_2, I). There exist non empty disjoint quasi open set A, in Y and qpI- open set B in Y s.t Y = $A \cup B$. Since Y is biopen in X and A and B are quasi open and qpI- open in X respectively and A and B are disjoint, then $qCl(A) \cap B = \emptyset = A \cap qpICl(B)$. This implies that A, B are qpI-separated sets in X. Thus, Y is not qpI-*s*-connected in (X, τ_1, τ_2, I). Hence we arrive at a contradiction and Y is qpI- connected in (X, τ_1, τ_2, I).
ii) $\Rightarrow$ i) Suppose Y is qpI-connected in (X, τ_1, τ_2, I) and Y is not qpI-*s*-connected in (X, τ_1, τ_2, I). There exist two qpI-separated sets A, B s.t $Y = A \cup B$. By Theorem 2.1, A and B are qpI-open and quasi open in Y respectively. Since Y is biopen in X, obviously A and B are qpI- open and quasi open in X respectively. Also Y is qpI-connected so Y cannot be written as the disjoint union of a nonempty quasi open set and a nonempty qpI- open set**.** This is a contradiction and Y is qpI-*s*-connected.

Theorem 2.5. Let (X, τ_1, τ_2, I) be an ideal bitopological space. If A is a qpI-*s*-connected set of X and H, G are qpI-separated sets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.
Proof: Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $(A \cap G) \cap qCl(A \cap H) \subset G \cap qpICl(H) = \emptyset$. By similar reasoning, we have $(A \cap H) \cap qCl(A \cap G) \subset H \cap qpICl(G) = \emptyset$. If $A \cap H$ and $A \cap G$ are nonempty, then A is not qpI- *s*-connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$ This implies that either $A \subset H$ or $A \subset G$.

Theorem 2.6. If A is a qpI-s-connected set of an ideal bitopological space (X, τ_1, τ_2, I) and $A \subset B \subset qCl(A) \cap qpICl(B)$ then B is qpI-s-connected.
Proof: The theorem can easily be proved by taking the contradiction.

Theorem 2.7. If $\{M_i: i \in I\}$ is a nonempty family of qpI-s-connected sets of an ideal bitopological space (X, τ_1, τ_2, I) with $\bigcap_{i\in I} Mi \neq \phi$ Then $\bigcup_{i\in I} Mi$ is qpI-s-connected.
Proof: Suppose that $\bigcup_{i\in I} Mi$ is not qpI-s-connected. Then we have $\bigcup_{i\in I} Mi = H \cup G$, where H and G are qpI-separated sets in X. Since $\bigcap_{i\in I} Mi \neq \phi$ we have a point x in $\bigcap_{i\in I} Mi$. Since $x \in \bigcup_{i\in I} Mi$, either $x \varepsilon H$ or $x \varepsilon G$. Suppose that $x \varepsilon H$. Since $x \varepsilon M_i$ for each $i \varepsilon I$, then M_i and H intersect for each $i \varepsilon I$. By theorem 2.5: $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \varepsilon I$ and hence $\bigcup_{i\in I} Mi \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \varepsilon G$. By similar way, we have that H is empty which is a contradiction. Thus, $\bigcup_{i\in I} Mi$ is qpI-s-connected.

Theorem 2.8. Suppose that $\{M_n: n \varepsilon N\}$ is an infinite sequence of qpI-connected open sets of an ideal space (X, τ_1, τ_2, I) and $M_n \cap M_{n+1 \neq} \phi$ for each $n \varepsilon N$. Then $\bigcup_{i\in I} Mi$ is qpI-s-connected.
Proof: By induction and Theorems 2.4 and 2.7, the set $P_n = \bigcup_{k \leq n} Mk$ is a qpI-connected open set for each $n \varepsilon N$. Also, P_n has a nonempty intersection. Thus $\bigcup_{n\in N} Mn$ is qpI-connected.

Definition 2.4. Let X be an ideal bitopological space (X, τ_1, τ_2, I) and $x \in X$. The union of all qpI-s-connected subsets of X containing x is called the qpI-component of X containing x.

Theorem 2.9. Each qpI-component of an ideal bitopological space (X, τ_1, τ_2, I) is a maximal qpI-s connected set of X.
Proof: Obvious.

Theorem 2.10. The set of all distinct qpI-components of an ideal bitopological space (X, τ_1, τ_2, I) forms a partition of X.

Proof: Let A and B be two distinct qpI-components of an ideal bitopological space (X, τ_1, τ_2, I) containing x and y respectively $\{x \neq y\}$. Suppose that A and B intersect. Then, by Theorem 2.7, $A \cup B$ is qpI-s-connected in X. Also, $A, B \subseteq A \cup B$, so A, B are not maximal and thus A, B are disjoint. Hence they partition X. by induction it can easily be proved that the set of all distinct qpI-components of X forms a partition of X.

Theorem 2.11. Each qpI- component of an ideal bitopological space (X, τ_1, τ_2, I) is qpI-closed in X.

Proof: Let A be a qpI-component of X. Therefore qpCl(A) is qpI-s-connected and A = qpCl (A). Thus, A is qpI-closed in X.

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