NANO TOPOLOGY VIA WEAK FORM OF OPEN SETS

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Abstract: The motive behind this article is to explore the possibility of the weak form of nano topology. By inducing the nano ω-open sets we evolve nano ω-open, nano pre-ω-open sets and nano contra ω-open sets. Finally using nano ω-open sets we discuss respective continuity as well as other related properties.

Keywords: Nano Topology, Nano α-Open, Nano ω-Open, Nano ω-Continuity.

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Introduction: Njastad [6], Levine [3] and Mashhour et al [5] respectively introduced the notions of α-open, semi-open and pre-open sets. Masshour [5] has also developed the concept of α-continuous mappings and α-open maps. The class of α-irresolute maps was introduced by Maheswari and Thakur [4]. Nano Topological space was introduced by M. Lellis Thivagar et.al [2] in the year 2012 can be described as a collection of nano open sets namely a non-empty finite universe and approximations for which the equivalence classes are building blocks. It is named as nano topology due to its size ie it has at most five open sets irrespective of the universe.

Preliminaries: We shall recall here a few basic concepts and definitions that will enable us to build up the proposed theory.

Definition 2.1 [7]: A ω-closed set if $cl(A) \subseteq G$ whenever $A \subseteq G$ and G is semi-open in (X, τ). The complement of a ω-closed set is called a ω-open set. The family of all ω-open set, denoted by τ_ω or ωO(X) is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Definition 2.2 [7]: A subset A of a space (X, τ) is said to be
1. semi-ω-open if $A \subseteq cl(int_\omega(A))$.
2. pre-ω-open if $A \subseteq int_\omega(cl(A))$.

3. α-ω-open if $A \subseteq int_{\omega}(cl(int_{\omega}(A)))$

Definition 2.3 [2]: Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and R be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$.

That is, $L_R(X) = \bigcup_{x\in\mathcal{U}}\{R(x): R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by X.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$.

That is,$U_R(X) = \bigcup_{x\in\mathcal{U}}\{R(x): R(x) \cap X \neq \emptyset\}$.

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as Xnor as not-$X$with respect to R and it is denoted by $B_R(X)$.

That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.4 [2]: Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$,where $X \subseteq \mathcal{U}$. $\tau_R(X)$ satisfies the following axioms:

1. $\mathcal{U}$ *and* $\emptyset \in \tau_R(X)$.
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of the elements of any finite sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

That is,$\tau_R(X)$forms a topology on $\mathcal{U}$ called the nano topology on $\mathcal{U}$ with respect to X. We call $(\mathcal{U}, \tau_R(X))$as the nano topological space. The elements of $\tau_R(X)$are called nano open sets.

Definition 2.5 [2]: If $(\mathcal{U}, \tau_R(X))$is a nano topological space with respect to X where $X \subseteq \mathcal{U}$ and if $A \subseteq \mathcal{U}$, then the nano interior of A is defined as the union of all nano open subsets of $A$and it is denoted by $\mathcal{N}int(A)$. That is, $\mathcal{N}int(A)$ is the largest nano-open subset of A. The nano closure of $A$is defined as the intersection of all nano closed sets containing $A$and it is denoted by $\mathcal{N}cl(A)$. That is, $\mathcal{N}cl(A)$ is the smallest nano closed set containing A.

Definition 2.6 [2]: Let $(\mathcal{U}, \tau_R(X))$be a nano topological space and $A \subseteq \mathcal{U}$. Then A is said to be

1. nano semi-open if $A \subseteq \mathcal{N}cl(\mathcal{N}int(A))$
2. nano pre-open if $A \subseteq \mathcal{N}int(\mathcal{N}cl(A))$
3. nano α-open if $A \subseteq \mathcal{N}int(\mathcal{N}cl(\mathcal{N}int(A))$

The complements of above nano open sets are called respective nano closed sets.

Characterizations: In this section we discuss about nano ω-open, nano semi-ω-open, nano pre-ω-open and establish their related properties.

Definition 3.1: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space $A \subseteq \mathcal{U}$. A is said to be nano ω-closed set if $\mathcal{N}cl(A) \subseteq H$ where $A \subseteq H$ and H is nano semi open. The complement of a nano ω-closed set is said to be nano ω-open. The family of all nano ω-open sets of a space $\big(\mathcal{U}, \tau_R(X)\big)$ is denoted by $\tau R\omega$ or $\mathcal{N}_{\omega}O(\mathcal{U}, X)$.

Example 3.2: Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \big\{\{a\}, \{c\}, \{b, d\}\big\}$. Let $X = \{a, b\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \big\{U, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\big\}$ which are nano open sets. $\mathcal{N}_{\omega}$-closed sets are
$\big\{U, \emptyset, \{c\}, \{c, d\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}\big\}$. $\mathcal{N}_{\omega}$-open sets are

$\{U, \emptyset, \{a\}, \{b\}, \{d\}, \quad \{a, d\}, \{b, d\}, \{a, b\}, \{a, b, d\}\}$.

Proposition 3.3: In a space $(\mathcal{U}, \tau_R(X))$, every nano open set is nano ω-open.
Proof: Let A be a nano open set in $\mathcal{U}$. Then A^C is a nano closed set. Therefore $\mathcal{N}cl(A^C) = A^C$. This implies $\mathcal{N}cl(A^C) \subseteq H$ whenever $A^C \subseteq H$ and H is a nano semi-open set in $(\mathcal{U}, \tau_R(X))$. Hence A^C is nano ω-closed or A is nano ω-open.

Theorem 3.4: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space with respect to X where $X \subseteq \mathcal{U}$. Let $A, B \subseteq \mathcal{U}$. Then

1. A is nano ω-closed if and only if $\mathcal{N}cl_\omega(A) = A$.
2. $A \subseteq \mathcal{N}cl_\omega(A) \subseteq \mathcal{N}cl(A)$.
3. $A \subseteq B \Rightarrow \mathcal{N}cl_\omega(A) \subseteq \mathcal{N}cl_\omega(B)$.
4. $\mathcal{N}cl_\omega(A \cup B) = \mathcal{N}cl_\omega(A) \cup \mathcal{N}cl_\omega(B)$
5. $\mathcal{N}cl_\omega(A \cap B) \subseteq \mathcal{N}cl_\omega(A) \cap \mathcal{N}cl_\omega(B)$

Proof: (i) If A is nano ω-closed, then A is the smallest nano ω-closed set containing $\mathcal{N}cl_\omega(A) = A$. Conversely, if $\mathcal{N}cl_\omega(A) = A$, then A is the smallest nano ω-closed set containing itself and hence A is nano ω-closed.
(ii) By definition of nano ω-closure, $A \subseteq \mathcal{N}cl_\omega(A)$. Next, $\mathcal{N}cl_\omega(A) \subseteq \mathcal{N}cl(A)$ if $x \in \mathcal{N}cl_\omega(A)$ and $x \notin \mathcal{N}cl(A)$, then $x \in \mathcal{U} - \mathcal{N}cl(A)$. That is $x \in \mathcal{N}int(\mathcal{U} - A)$. Therefore, there exists a nano open set G containing x such that $G \subseteq \mathcal{U} - A$. Since any nano open set is nano ω-open, G is a nano ω-open set containing x such that $G \subseteq \mathcal{U} - A$. That is $x \in \mathcal{N}int_\omega(U - A) = U - \mathcal{N}cl_\omega(A)$. Therefore, $x \notin \mathcal{N}cl_\omega(A)$, which is a contradiction. Therefore, $x \in \mathcal{N}cl_\omega(A)$ if $x \in \mathcal{N}cl_\omega(A)$. Thus, $A \subseteq \mathcal{N}cl_\omega(A) \subseteq \mathcal{N}cl(A)$.
(iii) when $A \subseteq B$, since $B \subseteq \mathcal{N}cl_\omega(B), A \subseteq \mathcal{N}cl_\omega(B)$. That is $\mathcal{N}cl_\omega(A)$ is a nano ω-closed set containing A. But $\mathcal{N}cl_\omega(A)$ is the smallest nano ω-closed set containing A. Therefore, $\mathcal{N}cl_\omega(A) \subseteq \mathcal{N}cl_\omega(B)$.
(iv) since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $\mathcal{N}cl_\omega(A) \subseteq \mathcal{N}cl_\omega(A \cup B)$ and $\mathcal{N}cl_\omega(B) \subseteq \mathcal{N}cl_\omega(A \cup B)$. Therefore, $\mathcal{N}cl_\omega(A) \cup \mathcal{N}cl_\omega(B) \subseteq \mathcal{N}cl_\omega(A \cup B)$. since $A \cup B \subseteq \mathcal{N}cl_\omega(A) \cup \mathcal{N}cl_\omega(B)$, and since $\mathcal{N}cl_\omega(A \cup B)$ is the smallest nano ω-closed set containing $A \cup B$, $\mathcal{N}cl_\omega(A \cup B) \subseteq \mathcal{N}cl_\omega(A) \cup \mathcal{N}cl_\omega(B)$. Thus, $\mathcal{N}cl_\omega(A \cup B) = \mathcal{N}cl_\omega(A) \cup \mathcal{N}cl_\omega(B)$.
(v) since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\mathcal{N}cl_\omega(A \cap B) \subseteq \mathcal{N}cl_\omega(A) \cap \mathcal{N}cl_\omega(B)$.

Theorem 3.5: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space with respect to X where $X \subseteq U$. Let $A, B \subseteq U$. Then
(i) A is nano ω-open if and only if $A = \mathcal{N}int_\omega(A)$.
(ii) $\mathcal{N}int_\omega(\mathcal{N}int_\omega(A)) = \mathcal{N}int_\omega(A)$
(iii) $A \subseteq B \Rightarrow \mathcal{N}int_\omega(A) \subseteq \mathcal{N}int(B)$.
(iv) $\mathcal{N}int_\omega(A \cap B) = \mathcal{N}int_\omega(A) \cap \mathcal{N}int_\omega(B)$
(v) $\mathcal{N}int_\omega(A) \cup \mathcal{N}int_\omega(B) \subseteq \mathcal{N}int_\omega(A \cup B)$

Example 3.6: $\mathcal{N}int_\omega(A \cup B) \neq \mathcal{N}int_\omega(A) \cup \mathcal{N}int_\omega(B)$. For example, let $\mathcal{U} = \{p, q, r, s\}$ with $\mathcal{U}/R = \{\{r\}, \{q, s\}, \{p\}\}$. Let $X = \{q, r\}$. Then $\tau_R(X) = \{U, \varphi, \{r\}, \{q, r, s\}, \{q, s\}\}$ and $\tau_R^\omega(X) = \{U, \varphi, \{r\}, \{q\}, \{s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{q, r, s\}\}$. If $A = \{p, q\}$ and $B = \{r, s\}$, then $\mathcal{N}int_\omega(A) = \{q\}$ and $\mathcal{N}int_\omega(B) = \{r, s\}$. Therefore, $\mathcal{N}int_\omega(A) \cup \mathcal{N}int_\omega(B) = \{q, r, s\}$. But $A \cup B = U$ and hence $\mathcal{N}int_\omega(A \cup B) = U$. Thus, $\mathcal{N}int_\omega(A \cup B) \neq \mathcal{N}int_\omega(A) \cup \mathcal{N}int_\omega(B)$.

Definition 3.7: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq U$. Then A is said to be:
(i) nano semi-ω-open if $A \subseteq \mathcal{N}cl(\mathcal{N}int_\omega(A))$.

(ii) nano pre-ω-open if $A \subseteq \mathcal{N}int_{\omega}(\mathcal{N}cl(A)$.
(iii) nano α-ω-open if $A \subseteq (\mathcal{N}int_{\omega}(\mathcal{N}cl(\mathcal{N}int_{\omega}(A)))$.
(iv) nano b-ω-open if $A \subseteq \mathcal{N}cl(\mathcal{N}int_{\omega}(A)) \cup \mathcal{N}int_{\omega}(\mathcal{N}cl(A))$.
The complements of the above are called the respective closed sets.

Proposition 3.8: In a space $(\mathcal{U}, \tau_R(X))$, every nano semi-open subset is nano semi-ω-open. **Proof:** Let H be nano semi-open in $(\mathcal{U}, \tau_R(X))$. Then $H \subseteq \mathcal{N}cl(\mathcal{N}int(A)) \subseteq \mathcal{N}cl(\mathcal{N}int_{\omega}(A))$. This proves that H is nano semi-ω-open.

Remark 3.9: The converse of the above Proposition is not true. For example, let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{c\}, \{b, d\}$ and $X = \{a, d\} \subseteq \mathcal{U}$. Then $\tau R(X) = \{\mathcal{U}, \emptyset, \{a\}, \{a, b, d\}, \{b, d\}\}$. $NSO(\mathcal{U}, X) = \{\mathcal{U}, \emptyset, \{a\}, \{a, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. $NS\omega O(\mathcal{U}, X) = \{\mathcal{U}, \emptyset, \{a\}, \{b\}, \{d\}, \{a, b\},$ $\{b, c\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}$. If $A = \{c, d\}$ is in nano semi-ω-open but is not in nano semi-open.

Comparisons: We shall now discuss about nano ω-continuous and nano contra ω-continuous.

Theorem 4.1: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space, then the following properties hold:
(i) Every nano ω-open set is nano α-ω-open.
(ii) Every nano α-ω-open set is nano pre-ω-open.
(iii) Every nano pre-ω-open set is nano b-ω-open.

Theorem 4.2: Let A be a subset of nano topological space $(\mathcal{U}, \tau_R(X))$. A is nano α-ω-open if and only if it is nano semi-ω-open and nano pre-ω-open.

Proposition 4.3: The intersection of a nano pre-ω-open set and nano open set is nano pre-ω-open.

Definition 4.4: A function $f: (\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano ω-continuous if $f^{-1}(B)$ is nano ω-open in $\mathcal{U}$ for every nano open B in $\mathcal{V}$.

Example 4.5: Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a, c\}, \{b\}, \{d\}\}$ and $X = \{c, d\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{d\}, \{a, c, d\}, \{a, c\}\}$. $\mathcal{N}_{\omega}O(\mathcal{U}, X) = \{\mathcal{U}, \emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. Let $\mathcal{V} = \{x, y, z, w\}$ with $\mathcal{V}/R' = \{\{x, z\}, \{y\}, \{w\}\}$ and $Y = \{z, w\}$. Then $\tau_{R'}(Y) = \{V, \emptyset, \{w\}, \{x, z, w\}, \{x, z\}\}$. Define $f: \mathcal{U} \to \mathcal{V}$ as $f(a) = x, f(c) = z, f(d) = w, f(b) = y$. Then $f^{-1}(\{w\}) = \{d\}$, $f^{-1}(\{x, z, w\}) = \{a, c, d\}$, $f^{-1}(\{x, z\}) = \{a, c\}$ and $f^{-1}(\{V\}) = \mathcal{U}$. That is, the inverse image of every nano open set in $\mathcal{V}$ is nano ω-open in $\mathcal{U}$. Therefore, f is nano ω-continuous.

Theorem 4.6: Every nano continuous function is nano ω-continuous.
Proof. Let $f: (\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_{R'}(Y))$ be nano continuous and B be nano open in V. Then $f^{-1}(B)$ is nano open in $\mathcal{U}$. Since any nano open set is nano ω-open, $f^{-1}(B)$ is nano ω-open in $\mathcal{U}$. Thus, inverse image of every nano open set is nano ω-open. Therefore, f is nano ω-continuous.

Theorem 4.7: A mapping $f: (\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano ω-continuous if and only if $f: (\mathcal{U}, \tau_R^{\omega}(Y)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano continuous.
Proof: A mapping $f: (\mathcal{U}, \tau_R(X)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano ω-continuous if and only if $f^{-1}(B)$ is nano ω-open for every nano open set B in V if and only if $f^{-1}(B) \in (\mathcal{U}, \tau_R^{\omega}(Y))$ for every B in $(\mathcal{V}, \tau_{R'}(Y))$ if and only if $f: (\mathcal{U}, \tau_R^{\omega}(Y)) \to (\mathcal{V}, \tau_{R'}(Y))$ is nano continuous.

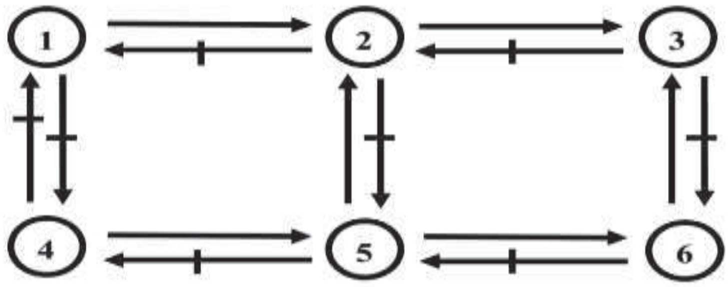
Definition 4.8: A function $f: (\mathcal{U}, \tau_R(X)) \rightarrow (\mathcal{V}, \tau_{R'}(Y))$ is nano contra ω-continuous if $f^{-1}(B)$ is nano ω-closed in $\mathcal{U}$ for each nano open B in $\mathcal{V}$.

Example 4.9: Let $\mathcal{U} = \{p, q, r\}$ with $\mathcal{U}/R = \{\{p\}, \{q, r\}\}$ and $X = \{p, r\} \subseteq \mathcal{U}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{p\}, \{q, r\}\}$. $\mathcal{N}\omega cl(U, X) = \{U, \emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\}$. Let $\mathcal{V} = \{a, b, c\}$ with $\mathcal{V}/R' = \{\{a, c\}, \{b\}\}$ and $Y = \{b\}$. Then $\tau_{R'}(Y) = \{\mathcal{V}, \emptyset, \{b\}\}$. Define $f : \mathcal{U} \rightarrow \mathcal{V}$ as $f(p) = a$, $f(q) = b$, $f(r) = c$. Then $f^{-1}(\{b\}) = \{q\}$ and $f^{-1}(\{\mathcal{V}\}) = \mathcal{U}$. That is, the inverse image of every nano open set in $\mathcal{V}$ is nano ω-closed in $\mathcal{U}$. Therefore, f is nano contra ω-continuous.

Theorem 4.10: A function $f: (\mathcal{U}, \tau_R(X)) \rightarrow (\mathcal{V}, \tau_{R'}(Y))$ then f is nano contra ω- continuous if and only if the inverse image of each nano closed set in $\mathcal{V}$, is nano ω-open in $\mathcal{U}$.

Proof: Let f be nano contra ω-continuous. Let B be a nano closed set in $\mathcal{V}$ and therefore B^C is nano open in $\mathcal{V}$. since f is nano contra ω-continuous, $f^{-1}(B^C)$ is nano closed in $\mathcal{U}$. But, $f^{-1}(B^C) = \{f^{-1}(B)\}^C$. Hence $f^{-1}(B)$ is nano ω-open in $\mathcal{U}$. Therefore, inverse image of each nano closed set in $\mathcal{V}$, is nano ω-open in $\mathcal{U}$. Conversely, Let B be an nano ω-open set in $\mathcal{V}$ then Bc is nano closed in $\mathcal{V}$. Since inverse image of each nano closed set in $\mathcal{V}$, is nano ω-open in $\mathcal{U}$, $f^{-1}(B^C)$ is nano ω-open in $\mathcal{U}$. Hence $f^{-1}(B)$ is nano closed in $\mathcal{U}$. Hence f is nano contra ω-continuous.

Remark 4.11: From the above discussions we get the following diagram. $1 \rightarrow 2$ represents 1 implies 2, $1 \nrightarrow 2$ represents 1 does not implies 2:



1. Nano open **2. Nano α-open** **3. Nano pre-open**
4. Nano ω-open **5. Nano αω-open** **6. Nano Pre ω-open**

Conclusion: In this paper, based on ω-open and closed sets, we have established some weak forms of nano topology, namely nano semi-ω-open, nano pre-ω-open, nano α-ω-open, nano β-ω-open sets etc. Also we studied their properties and established some results by using nano topology. Finally we made a comparative study and defined their relationship with one another that gives us an interesting result.

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