OPEN GEODETIC NUMBER OF THE LEXICOGRAPHIC PRODUCT OF GRAPHS

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Abstract: A subset $\boldsymbol{S}$ of vertices in a graph $\boldsymbol{G}$ is called a geodetic set if every vertex not in $\boldsymbol{S}$ lies in at least one interval between the vertices of $\boldsymbol{S}$. A geodetic set of minimum cardinality is a minimum geodetic set and this cardinality is the geodetic number, denoted by $\boldsymbol{g}(\boldsymbol{G})$. A set $\boldsymbol{S}$ of vertices of a connected graph $\boldsymbol{G}$ is an open geodetic set of $\boldsymbol{G}$ if for each vertex $\boldsymbol{v}$ in $\boldsymbol{G}$, either (i) $\boldsymbol{v}$ is a simplicial vertex of $\boldsymbol{G}$ and $\boldsymbol{v} \in \boldsymbol{S}$ or (ii) $\boldsymbol{v}$ is an internal vertex of a $\boldsymbol{x}-\boldsymbol{y}$ geodesic for some $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number, denoted by $\boldsymbol{og}(\boldsymbol{G})$. Upper bounds for the open geodetic number of lexicographic product graphs are proved and for several classes exact values are obtained.

Keywords: Lexicographic Product, Open Geodetic, Complete Graph.

Introduction: In Graph theory, the distance related problem has become the emerging field of applications. One kind of the distance variant is geodetic number problem. In recent years many researchers around the world has come up with numerous results in geodetic number and variance of it. Still many open problems and conjectures are yet to be answered. In 1993, Harary et al introduced the geodetic number of a graph [11] and the computation of the geodetic number is an NP-hard problem [12] for general graphs. The recent literature survey of this variant is studied in[13]. Geodetic number for the Cartesian product of graphs has been discussed in $[5-7]$ by various authors in different years. In particular the geodetic number of lexicographic product of graphs has been discussed in[8]. All graphs in this paper are simple, connected and finite. For other graph theoretical notations refer [1].

Preliminaries:
Definition 2.1: A vertex $u \in V(G)$ is said to be geodominated by the pair $\{x, y\}$ if u lies on some $x-y$ geodesic in G, for any $x, y \in V(G)$. The geodetic interval $I[x, y]$ consists of x, y together with all vertices geodominated by the pair $\{x, y\}$. If S is a set of vertices of G, then the geodetic closure $I[S]$ is the union of all sets $I[x, y]$ for $x, y \in S$. If $I[S] = V(G)$, then S is said to be a geodetic set of G. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set.

Definition 2.2: A vertex v of a graph G is called simplicial if its neighborhood $N(v)$ induces a clique of size equal to degree of v. Simplicial vertices are also referred to as extreme vertices.

Definition 2.3: [4] A set S of vertices in a connected graph G is an open geodetic set if for each vertex v in G, either (1) v is an extreme vertex of G and $v \in S$ or (2) v is an internal vertex of an $x-y$ geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number $og(G)$ of G.
Definition 2.4: [2] The lexicographic product of graphs G and H is the graph $G \circ H$ with the vertex set $V(G) \times V(H)$ and the edges (a, x) and (b, y) are adjacent if either $ab \in E(G)$ or $a = b$ and $xy \in E(H)$.

Remark 2.5: [3] The lexicographic product of graphs G and H is not always same as the lexicographic product of graphs H and G. i.e. $G \circ H \ncong H \circ G$. And it holds good for graphs G and H are both complete, G and H are both totally disconnected, there exists a graph K and integers $n \geq 1, m \geq 1$ such that $G = K^{\circ,n}$ and $H = K^{\circ,m}$ where $G^{\circ,n}$ denotes the n^{th} power of G with respect to the lexicographic product.

Theorem 2.6: [4] If a nontrivial connected graph G contains no simplicial vertices, then $og(G) \geq 4$

Main Results

Theorem 3.1: Let G and H be two graphs. Then $G \circ H$ contains simplicial vertices if and only if $H \cong K_n$ and G contains non empty set of simplicial vertices.

Proof: Let $V(G) = \{1,2,\ldots,m\}$ and $V(H) = \{1,2,\ldots,n\}$ such that $|V(G)| = m$ and $|V(H)| = n$. For $k \leq m$, let G contains k number of simplicial vertices and $S_G = \{x_1, x_2, \ldots, x_k\}$ be the set of simplicial vertices in G. Suppose that $H \cong K_n$. Then by the definition of the lexicographic product, the vertex set $S = \{(x_i, j) | x_i \in G, j \in H, 1 \leq i \leq k, 1 \leq j \leq n\}$ forms the set of simplicial vertices in $G \circ H$. Conversely, let us assume that $H \ncong K_n$ or G does not contain a simplicial vertex. In either case, any vertex $(x, y) \in V(G \circ H)$ will lie on a geodesic between $S_{G \circ H}$. Thus there are no simplicial vertices in $G \circ H$.

Corollary 3.2: Let G be a graph with m simplicial vertices. Then $og(G \circ K_n) \geq mn$.
Proof. By Theorem 3.1., $G \circ K_n$ contains mn number of simplicial vertices. Thus $og(G \circ K_n) \geq mn$. W

Theorem 3.3: Let P_n be a path graph on n vertices. Then $4 \leq og(P_m \circ P_n) \leq 6$, for $m, n \geq 4$.
Proof. $P_m \circ P_n$ does not contain a simplicial vertex by Theorem 3.1. Then $og(P_m \circ P_n) \geq 4$ [4]. Let *S={(1,1), (2,1),(2,n),(m-1,1),(m-1,n),(m,n)}*. Then the geodesics between the vertices (1,1) and (m, n) openly geodominates all the vertices except the vertices of the first and the last row. The geodesics between the vertices (2,1) and $(2, n)$ openly geodominates the vertices of the first row. And the geodesics between the vertices $(m-1,1)$ and $(m-1, n)$ openly geodominates the vertices of the last row. It is clear that all the vertices of $(P_m \circ P_n)$ lie on a geodesics between S.

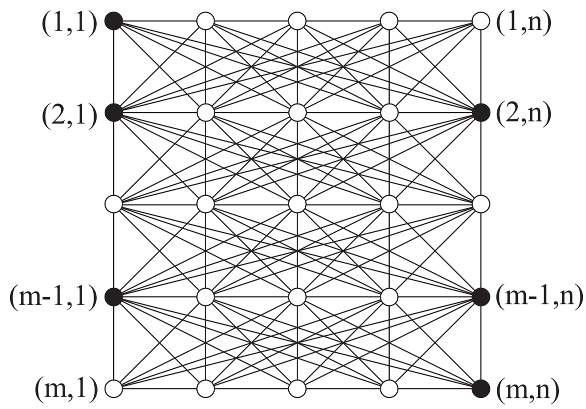

Figure 1: $P_5 \circ P_5$

Theorem 3.4. Let C_n be a cycle graph on n vertices. Then $4 \leq og(P_m \circ C_n) \leq 6$, for $m, n \geq 4$.
Proof. $P_m \circ C_n$ does not contain a simplicial vertex by Theorem 3.1. Then $og(P_m \circ C_n) \geq 4$[4]. Let *S={(1,1),(2,2),(2,n-1),(m-1,2),(m-1,n-1),(m,n)}*. Then the geodesics between the vertices (1,1) and (m, n) openly geodominates all the vertices except the vertices of the first and the last row. The geodesics between the vertices (2,2) and $(2, n-1)$ openly geodominates the vertices of the first row. And the geodesics between the vertices $(m-1,2)$ and $(m-1, n-1)$ openly geodominates the vertices of the last row. It is clear that all the vertices of $(P_m \circ C_n)$ lie on a geodesics between S.

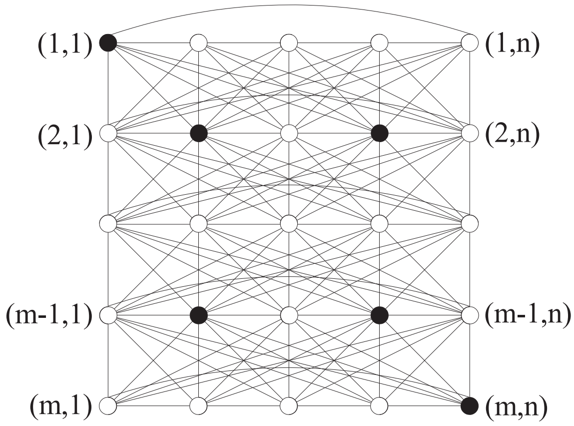

Figure 2: $P_5 \circ C_5$

Theorem 3.5: $og(P_n \circ K_2) = 4$.
Proof: Denote the vertices of P_n as $1,2,\ldots,n$ and K_2 as a,b. Then $V(P_n \circ K_2)$=*{(1,a),(1,b),(2,a),(2,b),…,(n,a),(n,b)})*. In which $(1,a),(1,b),(n,a)$ and (n,b) are simplicial vertices. These vertices forms a unique open geodetic set for $P_n \circ K_2$. Therefore $og(P_n \circ K_2) = 4$. W

Theorem 3.6: $og(K_2 \circ P_n) = 4$.
Proof: We label the graph as in the previous theorem. Then $(V(K_2 \circ P_n) = \{(a,1),(b,1),(a,2),\ (b,2),\ldots,(a,n),(b,n)\})$.Let $S = \{(a,1),(b,1),(a,n),(b,n)\}$. Clearly S openly geodominates the vertices of $P_n \circ K_2$. Therefore $og(P_n \circ K_2) \leq q4$. And exclusion of any one of the vertex from S will not form a geodetic set. Thus $og(K_2 \circ P_n) = 4$. W

Remark 3.7: In fact $P_n \circ K_2 \ncong K_2 \circ P_n$, we have $og(P_n \circ K_2) = og(K_2 \circ P_n) = 4$. There are certain graphs in which $G \circ H \ncong H \circ G$, but $og(G \circ H) = og(H \circ G)$.

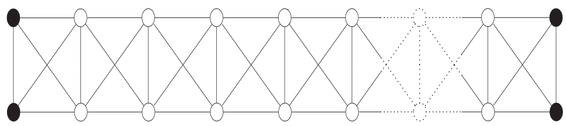

Figure 3: $P_n \circ K_2$

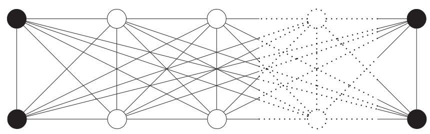

Figure 4: $K_2 \circ P_n$

Theorem 3.8: The lexicographic product of complete graphs is a complete graph.
Proof: Let K_m be the complete graph on m vertices. By the definition of lexicographic product, $|V(K_m \circ K_n)| = mn$ and $|E(K_m \circ K_n)| = \frac{mn(mn-1)}{2}$. This is possible only if $K_m \circ K_n$ is a complete graph. Thus the lexicographic product of complete graphs is a complete graph. W

Theorem 3.9: $og(K_m \circ K_n) = mn$.
Proof: By theorem 3.8, $K_m \circ K_n$ is a complete graph on mn vertices. Therefore $og(K_m \circ K_n) = mn$.

Conclusion: In this paper we proved the lower bound for the special classes of the open geodetic number of the lexicographic product of graphs.

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