DIVISOR DEGREE INDEX OF GRAPHS USING GRAPH OPERATIONS

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Abstract: In this paper, we introduce the concepts of equivalent divisor degree of graphs G1 and G2. Also, we obtain the sharp upper bound for divisor degree index of a simple graph G and also find divisor degree index for certain graphs by using some graph operations.

Keywords: Degree, Divisor Degree of A Vertex, Divisor Degree Index, Equivalent Divisor Degree.

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1. Introduction: Let G be a simple graph with n vertices and m edges. Let di be the degree of a vertex vi.

Definition 1.1: [4]: For a graph G, the divisor degree of a vertex vi denoted by dd(vi) or ddi , is defined as

$$dd(v_i) = \begin{cases} \sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right), if\ v_i\ and\ v_k\ are\ adjacent \\ \quad 1 \qquad\qquad , if\ d_i = d_k;\ v_i\ and\ v_k\ are\ adjacent \\ \quad 0 \qquad\qquad , otherwise \end{cases}$$

where $[x]$ denotes integral part of real number x and $\sum_{i\sim k}$ means summation over all pair of adjacent vertices v_i and v_k .

Definition 1.2: [4]: For a graph G, the divisor degree index dd(G) is defined as

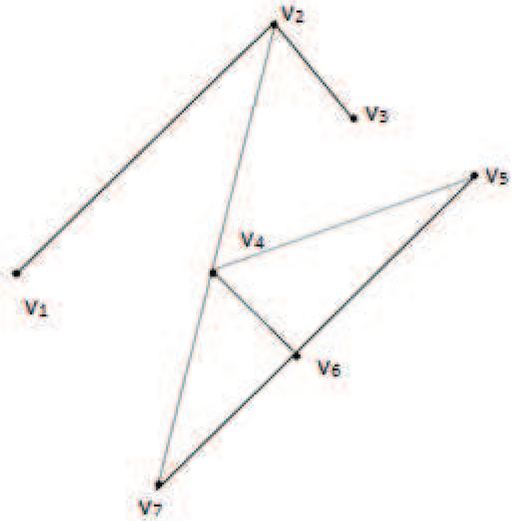
$$dd(G) = \sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)\right) = \sum_{i=1}^{n} dd(v_i)$$

Definition 1.3: [4]: For any graph G, we define as follows:

$\delta_{dd} = min\{dd(v)/v \in V(G)\}$ is called minimum divisor degree of G.

$\Delta_{dd} = max\{dd(v)/v \in V(G)\}$ is called maximum divisor degree of G.

Example 1.4: Consider a graph G



From the graph G, we have

$$dd(v_1) = \left(\left[\frac{d_1}{d_2}\right] + \left[\frac{d_2}{d_1}\right]\right) = 3.$$

$$dd(v_2) = \left(\left[\frac{d_2}{d_1}\right] + \left[\frac{d_1}{d_2}\right]\right) + \left(\left[\frac{d_2}{d_3}\right] + \left[\frac{d_3}{d_2}\right]\right) + \left(\left[\frac{d_2}{d_4}\right] + \left[\frac{d_4}{d_2}\right]\right) = 7.$$

Similarly, $dd(v_3) = 3, dd(v_4) = 6, dd(v_5) = 3, dd(v_6) = 3$ and $dd(v_7) = 3.$

$$dd(G) = 3 + 7 + 3 + 6 + 3 + 3 + 3 = 28.$$

$$\delta_{dd}(G) = min\{3, 7, 3, 6, 3, 3, 3\} = 3.$$

$$\Delta_{dd}(G) = max\{3, 7, 3, 6, 3, 3, 3\} = 7.$$

With this idea, we define a new concept named as equivalent divisor degree of graphs G1 and G2 .

Definition 1.5: If the graphs G1 and G2 have the same divisor degree index, then G1and G2 are said to be equivalent divisor degree.

Observation 1.6: The divisor degree index of r -regular graph is equal to sum of the degree of the vertices of a graph G (that is twice the number of edges). Note that the first theorem of graph theory on divisor degree is true for r-regular graph. In example 1.4, $dd(G) = 28 = 2 \times 8$, we have the following results.

Result 1.7: [4]:
dd(Cn) = 2m.
dd(Kn) = 2m.
The following well known and related bounding results are needed for the later part of this paper.

Theorem 1.8: [4]: If P_n is a path graph of order n, then $dd(P_n) = 2(n + 1)$ $(n \geq 3)$. 3.

Lemma 1.9: [4]: Let G be a simple graph with n vertices and m edges. Then $dd(G) \geq 2m$ with equality holds if G is regular.

2. Divisor Degree Index for Some Graphs: In this section, we obtain the sharp upper bound for divisor degree index of a simple graph G and also find divisor degree index for certain graphs by using some graph operations such as join, corona, complement, subdivision of graph, etc.,

Theorem 2.1: If G is a connected graph with n vertices and m edges, then

$$2m \leq dd(G) < n(n^2 - 2n + 2).$$

Proof: By definition 1.1, we have

$$\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right] < \left(\frac{d_i}{d_k} + \frac{d_k}{d_i}\right)$$
$$\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right) < (n-1)\left(\frac{d_i}{d_k} + \frac{d_k}{d_i}\right)$$
$$\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)\right) < \sum_{i=1}^{n}(n-1)\left(\frac{d_i}{d_k} + \frac{d_k}{d_i}\right)$$
$$\sum_{i=1}^{n} dd(v_i) < \sum_{i=1}^{n}(n-1)\left(n-1+\frac{1}{n-1}\right)$$
$$dd(G) < n(n^2-2n+2) \qquad (1)$$

By lemma 1.9 and equation (1), we have
$$2m \leq dd(G) < n(n^2-2n+2)$$

Theorem 2.2: If $F_{1,n-1}$is a fan graph of order n (n > 2), then
$$dd(F_{1,n-1}) = 2\left(n + (n-3)\left[\frac{n-1}{3}\right] + 2\left[\frac{n-1}{2}\right] - 2\right)$$
Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of $F_{1,n-1} = K_1 + P_{n-1}$ and di be the degree of vi . Here $v_1, v_2, \dots, v_{n-1}$ are the vertices of Pn−1 (n > 2) and so vn is adjacent to $v_1, v_2, \dots, v_{n-1}$. We have $d(v_1) = d(v_{n-1}) = 2, dd(v_n) = n-1$ and $dd(v_i) = 3\ for\ i = 2,3,\dots,n-2$.
Now, $dd(v_1) = \left(\left[\frac{d_1}{d_n}\right] + \left[\frac{d_n}{d_1}\right]\right) + \left(\left[\frac{d_1}{d_2}\right] + \left[\frac{d_2}{d_1}\right]\right) = \left[\frac{n-1}{2}\right] + 1$.
Also, $dd(v_{n-1}) = \left[\frac{n-1}{2}\right] + 1$.
$dd(v_2) = \left(\left[\frac{d_2}{d_1}\right] + \left[\frac{d_1}{d_2}\right]\right) + \left(\left[\frac{d_2}{d_3}\right] + \left[\frac{d_3}{d_2}\right]\right) + \cdots + \left(\left[\frac{d_2}{d_n}\right] + \left[\frac{d_n}{d_2}\right]\right) = 1 + 1 + \left[\frac{n-1}{3}\right]$.
Also, $dd(v_{n-2}) = 2 + \left[\frac{n-1}{3}\right]$.
Thus $dd(v_j) = 2 + \left[\frac{n-1}{3}\right]\ for\ j = 3,4,\dots,n-3$ and
$dd(v_n) = 2\left[\frac{n-1}{2}\right] + (n-3)\left[\frac{n-1}{3}\right]$.
Then $dd(F_{1,n-1}) = \sum_{k=1}^{n} v_k = 2\left(\left[\frac{n-1}{2}\right] + 1\right) + 2\left(2 + \left[\frac{n-1}{3}\right]\right) + (n-5)\left(2 + \left[\frac{n-1}{3}\right]\right) + 2\left(2\left[\frac{n-1}{2}\right] + (n-3)\left[\frac{n-1}{3}\right]\right)$

Therefore, $dd(F_{1,n-1}) = 2\left(n + (n-3)\left[\frac{n-1}{3}\right] + 2\left[\frac{n-1}{2}\right] - 2\right)$.
Now, we obtain the divisor degree index of corona of some standard graphs.

Definition 2.3: Comb is a graph obtained by joining a single pendent edge to each vertex of a path. It is denoted by $P_p \circ K_1$.

Theorem 2.4: Let $P_p \circ K_1$ be a comb graph with n = 2p vertices. Then
$dd(P_p \circ K_1) = 2(4p-3)$.
Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p}$ be the vertices of $P_p \circ K_1$and d_i be the degree of v_i . Let $v_1, v_2, \dots, v_p$ be the vertices of P_p . Choose v_l is adjacent to v_{p+l} for $l = 1,2,\dots,p$.
Then $d(v_1) = d(v_p) = 2, d(v_i) = 3\ for\ i = 2,3,\dots,p-1$ and
$d(v_j) = 1\ for\ j = p+1, p+2, \dots, 2p$.
Now, $dd(v_1) = \left(\left[\frac{d_1}{d_{p+1}}\right] + \left[\frac{d_{p+1}}{d_1}\right]\right) + \left(\left[\frac{d_1}{d_2}\right] + \left[\frac{d_2}{d_1}\right]\right) = 2 + 1 = 3$.
Also, $dd(v_p) = 3$.
$dd(v_2) = \left(\left[\frac{d_2}{d_1}\right] + \left[\frac{d_1}{d_2}\right]\right) + \left(\left[\frac{d_2}{d_3}\right] + \left[\frac{d_3}{d_2}\right]\right) + \left(\left[\frac{d_2}{d_{p+2}}\right] + \left[\frac{d_{p+2}}{d_2}\right]\right) = 1 + 1 + 3 = 5$.
Thus $(v_i) = 5\ for\ i = 2,3,\dots,p-1$, $dd(v_{p+1}) = \left(\left[\frac{d_{p+1}}{d_1}\right] + \left[\frac{d_1}{d_{p+1}}\right]\right) = 2$,
$dd(v_k) = 3\ for\ k = p+2, p+3, \dots, 2p-1$ and $dd(v_{2p}) = 2$.

Then $dd(P_p\circ K_1) = \sum_{m=1}^{2p} v_m = 2\times 3 + 5(p-2) + 2\times 2 + 3(p-2)$

$$dd(P_p\circ K_1) = 2(4p-3).$$

Theorem 2.5: Let $C_p\circ K_1$ be corona of cycle graph with n = 2p vertices. Then $dd(C_p\circ K_1) = 8p$.
Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p}$ be the vertices of $C_p\circ K_1$and d_i be the degree of v_i . Let $v_1, v_2, \dots, v_p$ be the vertices of C_p . Choose v_i is adjacent to v_{p+i} for $i = 1,2,\dots,p$.
Then $d(v_i) = 3\ for\ i = 1,2,\dots,p$ and $d(v_j) = 1\ for\ j = p+1, p+2, \dots, 2p$.
Now $dd(v_i) = 5\ for\ i = 1,2,\dots,p$ and $dd(v_j) = 3\ for\ j = p+1, p+2, \dots, 2p$.
Hence, $dd(C_p\circ K_1) = \sum_{i=1}^{2p} v_i = 5\times p + 3\times p = 8p$.

Theorem 2.6: Let $W_p\circ K_1$ be corona of wheel graph with n = 2p vertices. Then $dd(W_p\circ K_1) = 2(7p - \left[\frac{p}{4}\right] - 10)$.
Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p}$ be the vertices of $W_p\circ K_1$and d_i be the degree of v_i . Let $v_1, v_2, \dots, v_p$ be the vertices of W_p . Choose v_i is adjacent to v_{p+i} for $i = 1,2,\dots,p$. Then $d(v_i) = 4\ for\ i = 1,2,\dots,p$ and $d(v_j) = 1\ for\ j = p+1, p+2, \dots, 2p$.
Now,$dd(v_k) = 6 + \left[\frac{p}{4}\right]\ for\ k = 1,2,\dots,p-1$;$dd(v_p) = p + (p-1)\left[\frac{p}{4}\right]$; $dd(v_l) = 4\ for\ l = p+1, p+2, \dots, 2p-1$ and $dd(v_{2p}) = p$.

Hence, $dd(W_p\circ K_1) = \sum_{m=1}^{2p} v_m = (p-1)\left(6 + \left[\frac{p}{4}\right]\right) + p + (p-1)\left[\frac{p}{4}\right] + 4(p-1) + p$.
Therefore, $dd(W_p\circ K_1) = 2\left(7p - \left[\frac{p}{4}\right] - 10\right)$.

Theorem 2.7: Let $K_{1,p-1}\circ K_1$ be corona of star graph with n = 2p vertices. Then $dd(K_{1,p-1}\circ K_1) = 2(3p + (p-1)\left[\frac{p}{2}\right] - 2)$.
Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p}$ be the vertices of $K_{1,p-1}\circ K_1$and d_i be the degree of v_i . Let $v_1, v_2, \dots, v_p$ be the vertices of $K_{1,p-1}$. Choose v_i is adjacent to v_{p+i} for $i = 1,2,\dots,p$. Then $d(v_i) = 2\ for\ i = 1,2,\dots,p-1$; $d(v_p) = p$ and $d(v_j) = 1\ for\ j = p+1, p+2, \dots, 2p$.
Now $dd(v_k) = 2 + \left[\frac{p}{2}\right]\ for\ k = 1,2,\dots,p-1$;$dd(v_p) = p + (p-1)\left[\frac{p}{2}\right]$; $dd(v_l) = 2\ for\ l = p+1, p+2, \dots, 2p-1$ and $dd(v_{2p}) = p$.
Hence, $dd(K_{1,p-1}\circ K_1) = \sum_{m=1}^{2p} v_m = (p-1)\left(2 + \left[\frac{p}{2}\right]\right) + p + (p-1)\left[\frac{p}{2}\right] + 2(p-1) + p$.
Therefore, $dd(K_{1,p-1}\circ K_1) = 2\left(3p + (p-1)\left[\frac{p}{2}\right] - 2\right)$.

Theorem 2.8: Let $(K_p)^+$be thorn graph of complete graph of order n = 2p. Then $dd(K_p)^+ = p(3p-1)$.
Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p}$ be the vertices of $(K_p)^+$ of order n = 2p and d_i be the degree of v_i . Let $v_1, v_2, \dots, v_p$ be the vertices of K_p , then each i-vertex is adjacent to every $(p-1)$ vertices for $i = 1,2,\dots,p$. Further v_i is adjacent to v_{p+i} for $i = 1,2,\dots,p$. Then $d(v_i) = p\ for\ i = 1,2,\dots,p$ and $d(v_j) = 1\ for\ j = p+1, p+2, \dots, 2p$.

$$\text{Now, } dd(v_1) = \left(\left[\frac{d_1}{d_{p+1}}\right] + \left[\frac{d_{p+1}}{d_1}\right]\right) + \left(\left[\frac{d_1}{d_2}\right] + \left[\frac{d_2}{d_1}\right]\right) + \left(\left[\frac{d_1}{d_3}\right] + \left[\frac{d_3}{d_1}\right]\right) + \cdots + \left(\left[\frac{d_1}{d_p}\right] + \left[\frac{d_p}{d_1}\right]\right) = p + (p-1).$$

Then, $dd(v_i) = 2p - 1\ for\ i = 1,2,\dots,p$.
Also, $dd(v_j) = p\ for\ j = p+1, p+2, \dots, 2p$.
Hence, $dd\left((K_p)^+\right) = \sum_{k=1}^{2p} v_k = (2p-1)\times p + p^2 = p(3p-1)$.
Now, we obtain the divisor degree index of complement of some standard graphs.

Theorem 2.9: Let $\overline{C_n}$ be the complement of C_n of order n (n > 3). Then $dd(\overline{C_n}) = n(n-3)$.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of $\overline{C_n}$, (n > 3) .
Then, $\overline{C_n}$ is (n – 3) – regular graph, that is, $dd(v_i) = n - 3\ for\ i = 1,2, \dots, n$. Hence $dd\left(\overline{C_n}\right) = n(n-3)$.

Theorem 2.10: Let $\overline{P_n}$ be the complement of P_n of order n (n > 4).
Then $dd\left(\overline{P_n}\right) = (n-1)(n-2)$.
Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of $\overline{P_n}$, (n > 4) .
Then, $dd(v_1) = dd(v_n) = n - 2$ and $dd(v_i) = n - 3\ \ for\ i = 2,3, \dots, n-1$.
Hence, $dd\left(\overline{P_n}\right) = (n-1)(n-2)$.

Theorem 2.11: Let $\overline{K_{1,n-1}}$ be the complement of $K_{1,n-1}$ of order n (n > 2).
Then $dd\left(\overline{K_{1,n-1}}\right) = (n-1)(n-2)$.
Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of $\overline{K_{1,n-1}}$, (n > 2) .
Then, $dd(v_i) = n - 2\ for\ i = 1,2, \dots, n-1$ and $dd(v_n) = 0$.
Hence, $dd\left(\overline{K_{1,n-1}}\right) = (n-1)(n-2)$.

Observation 2.12: By theorem 2.10 and 2.11, we observe that $\overline{P_n}$ and $\overline{K_{1,n-1}}$ have the same divisor degree index. Then $\overline{P_n}$ and $\overline{K_{1,n-1}}$ are said to be equivalent divisor degree.

Theorem 2.13: Let $\overline{W_n}$ be the complement of W_n of order n (n > 4). Then $dd\left(\overline{W_n}\right) = (n-1)(n-4)$.
Proof: Let $v_1, v_2, \dots, v_n$ be the vertices of $\overline{W_n}$, (n > 4) .
Then, $dd(v_i) = n - 4\ for\ i = 1,2, \dots, n-1$ and $dd(v_n) = 0$.
Hence, $dd\left(\overline{W_n}\right) = (n-1)(n-4)$.

Theorem 2.14: Let $S(P_n)$ be the subdivision of path graph of order $2n - 1$. Then $dd\big(S(P_n)\big) = 4n$.
Proof: By theorem 1.8, we have $dd(P_n) = 2(n+1)$.
Then $dd\big(S(P_n)\big) = 2(2n - 1 + 1) = 4n$.

Theorem 2.15: Let $S(C_n)$ be the subdivision of cycle graph of order $2n$. Then $dd\big(S(C_n)\big) = 4n$.
Proof: By theorem 1.7, we have $ddC_n) = 2n$. Then $dd\big(S(C_n)\big) = 4n$.

Observation 2.16: By theorem 2.14 and 2.15, we observe that $S(P_n)$ and $S(C_n)$ have the same divisor degree index. Then $S(P_n)$ and $S(C_n)$ are said to be equivalent divisor degree.

Theorem 2.17: Let $F_3^{(p)}$ be friendship graph of order $n = 2p + 1$, where (p) denotes the number of copies of K_3 . Then $dd(F_3^{(p)}) = 2p(2p+1)$.
Proof: Let $v_1, v_2, \dots, v_{2p}, v_{2p+1}$ be the vertices of $F_3^{(p)}$of order $n = 2p + 1$, where (p) denotes the number of copies of K_3 and d_i be the degree of v_i respectively . Choose v_k is adjacent to v_{k+1} and v_{2p+1} for $k = 1,3, \dots, 2p-1$; v_{2p+1} is adjacent to v_i for $i = 2,4, \dots, 2p$ Then $d(v_j) = 2\ for\ j = 1,2, \dots, 2p$ and $d(v_{2p+1}) = 2p$.
Now, $dd(v_1) = \left(\left[\frac{d_1}{d_2}\right] + \left[\frac{d_2}{d_1}\right]\right) + \left(\left[\frac{d_1}{d_{2p+1}}\right] + \left[\frac{d_{2p+1}}{d_1}\right]\right) = 1 + p$.
Thus, $dd(v_i) = p + 1\ for\ i = 1,2, \dots, 2p$ and
$$dd(v_{2p+1}) = \left(\left[\frac{d_{2p+1}}{d_1}\right] + \left[\frac{d_1}{d_{2p+1}}\right]\right) + \dots + \left(\left[\frac{d_{2p+1}}{d_{2p}}\right] + \left[\frac{d_{2p}}{d_{2p+1}}\right]\right) = 2p^2.$$
Hence, $dd(F_3^{(p)}) = \sum_{i=1}^{2p+1} v_i = (p+1) \times 2p + 2p^2 = 2p(2p+1)$.

Definition 2.18: The windmill graph $K_n^{(m)}$ is defined as only one centre vertex, by taking m copies of the complete graph K_n.The number of vertices are
$N = m(n-1) + 1$ and the number of edges are $M = \frac{mn(n-1)}{2}$.

Theorem 2.19: Let $K_n^{(m)}$ be windmill graph of order $N = m(n-1)+1$, where (m) denotes number of copies of K_n .Then $dd(K_n^{(m)}) = m(n-1)(2m+n-2)$.

Proof: Let $v_1, v_2, \dots, v_N$ be the vertices of $K_n^{(m)}$of order $N = m(n-1)+1$, where (m) denotes the number of copies of K_n and d_i be the degree of v_i respectively . Choose $v_1, v_2, \dots, v_n$ as the vertices of K_n, then each i – vertex is adjacent to every $(n-1)$ vertices for $i = 1,2,\dots,n$. Similarly for other $m-1$ copies of K_n.

Likewise v_N is adjacent to v_i for $i = 1,2,\dots,N-1$. Then $d(v_i) = n-1\ for\ i = 1,2,\dots,N-1$ and $d(v_N) = N-1$. Now,

$$dd(v_1) = \left(\left[\frac{d_1}{d_N}\right] + \left[\frac{d_N}{d_1}\right]\right) + \left(\left[\frac{d_1}{d_2}\right] + \left[\frac{d_2}{d_1}\right]\right) + \cdots + \left(\left[\frac{d_1}{d_{n-1}}\right] + \left[\frac{d_{n-1}}{d_1}\right]\right) = \left[\frac{N-1}{n-1}\right] + (n-2).$$

Thus, $dd(v_i) = \left[\frac{N-1}{n-1}\right] + (n-2) for\ i = 1,2,\dots,N-1$ and

$$dd(v_N) = \left(\left[\frac{d_N}{d_1}\right] + \left[\frac{d_1}{d_N}\right]\right) + \cdots + \left(\left[\frac{d_N}{d_{N-1}}\right] + \left[\frac{d_{N-1}}{d_N}\right]\right) = \left[\frac{N-1}{n-1}\right](N-1).$$

Hence,

$$dd(K_n^{(m)}) = \sum_{i=1}^{N} v_i = (N-1)\left(\left[\frac{N-1}{n-1}\right] + (n-2)\right) + \left[\frac{N-1}{n-1}\right](N-1)$$
$$= m(n-1)(2m+n-2).$$

Definition 2.20: The (p, q) -lollipop graph is a special type of graph consisting of a complete graph(clique) on p vertices and a path graph on q vertices, connected with a bridge having p + q vertices and $\binom{m}{2} + n$ edges. It is denoted by $L_{p,q}$.

Theorem 2.21: Let $L_{p,q}$ be lollipop graph with p + q vertices. Then

$$dd(L_{p,q}) = p(p-1) + 2\left[\frac{p}{2}\right] + 2q.$$

Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{p+q}$ be the vertices of $L_{p,q}$ of order $p+q$and d_i be the degree of v_i respectively. Choose $v_1, v_2, \dots, v_p$ as the vertices of K_p; $v_{p+1}, v_{p+2}, \dots, v_{p+q}$ as the vertices of P_q and $v_p v_{p+1}$ be a bridge. Since $v_1, v_2, \dots, v_p$ are the vertices of K_p; then each i – vertex is adjacent to every $(p-1)$ vertices for $i = 1,2,\dots,p$. Also v_j is adjacent to v_{j+1} for $j = p+1, p+2, \dots, p+q-1$. Then $d(v_i) = p - 1\ for\ i = 1,2,\dots,p-1$; $d(v_p) = p$; $d(v_j) = 2\ for\ j = p+1, p+2, \dots, p+q-1$ and $d(v_{p+q}) = 2$.

Now, $dd(v_i) = p-1\ for\ i = 1,2,\dots,p-1; dd(v_p) = (p-1)\left[\frac{p-1}{p}\right] + \left[\frac{p}{2}\right]$;

$dd(v_{p+1}) = \left[\frac{p}{2}\right] + 1$; $dd(v_k) = 2\ for\ k = p+2, p+3, \dots, p+q-2$;

$dd(v_{p+q-1}) = 3$ and $dd(v_{p+q}) = 2$. Hence,

$$dd(L_{p,q}) = (p-1)^2 + (p-1) + \left[\frac{p}{2}\right] + \left[\frac{p}{2}\right] + 1 + 2(q-3) + 3 + 2$$
$$= p(p-1) = 2\left[\frac{p}{2}\right] + 2q.$$

Definition 2.22: The (p, q) -tadpole graph is a special type of graph consisting of a cycle graph on m (at least 3) vertices and a path graph on q vertices, connected with a bridge having p + q vertices and p + q edges. It is denoted by $T_{p,q}$.

Theorem 2.23: Let $T_{p,q}$ be tadpole graph with p + q vertices. Then

$$dd(T_{p,q}) = 2(p+q+1).$$

Proof: Let $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{p+q}$ be the vertices of $T_{p,q}$of order $p+q$ and d_i be the degree of v_i respectively. Choose $v_1, v_2, \dots, v_p$ as the vertices of C_p; $v_{p+1}, v_{p+2}, \dots, v_{p+q}$ as the vertices of P_q and $v_p v_{p+1}$ be a bridge. Choose v_i is adjacent to v_{i+1} for $i = 1,2,\dots,p-1$; v_p is adjacent to v_1, v_{p+1} and v_j is adjacent to v_{j+1} for $j = p+1, p+2, \dots, p+q-1$ Then $d(v_i) = 2\ for\ i = 1,2,\dots,p-1$; $d(v_p) = 3, d(v_j) = 2\ for\ j = p+1, p+2, \dots, p+q-1$ and $d(v_{p+q}) = 1$.

Now, $dd(v_i) = 2\ for\ i = 1,2,\dots,p-1; dd(v_p) = 3$;

$dd(v_{p+q-1}) = 3$; $dd(v_k) = 2\ for\ k = p+1, p+2, \dots, p+q-2$;

and $dd(v_{p+q}) = 2$. Hence,

$dd(L_{p,q}) = 2(p-1)+3+2(q-2)+3+2 = 2(p+q+1)$.

Definition 2.24: The n -barbell graph is a special type of undirected graph consisting of two non-overlapping n -vertex cliques together with a single edge that has an endpoint in each clique having 2n vertices and $2\binom{n}{2}+1$ edges.

Theorem 2.25: The divisor degree index of n -barbell graph with order 2n is 2(n2 – n + 1).

Proof: Let $v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}$ be the vertices of n- barbell graph of order $2n$ and d_i be the degree of v_i respectively. Let $(v_1, v_2, \dots, v_n)$ and $(v_{n+1}, v_{n+2}, \dots, v_{2n})$ be n- vertex cliques and $v_n v_{n+1}$ be a bridge. Choose each i-vertex is adjacent to every $(n-1)$ vertices for $i = 1,2,\dots,n$. Choose v_n is adjacent to v_i for $i = 1,2,\dots,n-1$; v_n is adjacent to v_{n+1} and v_{n+1} is adjacent to v_j for $j = n+2, n+3, \dots, 2n$.

Then $dd(v_i) = n-1\ for\ i = 1,2,\dots,n-1$; $dd(v_n) = n, dd(v_{n+1}) = n, dd(v_j) = n-1\ for\ j = n+2, n+3, \dots, 2n$.

Hence the divisor degree index of n-barbell graph

$$= (n-1)^2 + 2n + (n-1)^2 = 2(n^2 - n + 1).$$

Definition 2.26: [1]: For a connected graph G, R(G) is obtained from G by adding a new vertex corresponding to each edge of G , then joining each new vertex to the end vertices of the corresponding edge.

Theorem 2.27: Let $R(P_p)$ be a graph of order $2p-1$, where p is the number of vertices of path graph Pp (p > 2) . Then $dd\left(R(P_p)\right) = 10(p-1)$.

Proof: Let $v_1, v_2, \dots, v_p$ be the vertices of P_p $(p > 2)$ with $p-1$ edges. Introduce a new vertex corresponding to $p-1$ edges and then joining each new vertex corresponding edge, we get $R(P_p)$ graph of order $2p-1$ $(p > 2)$ say $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p-1}$ and $d_i$is the degree of v_i. Since $v_1, v_2, \dots, v_p$are the vertices of P_p, v_i is adjacent to v_{i-1}, v_{i+1} for $i = 2,3,\dots,p-1$. Choose v_{p+j} is adjacent to v_j, v_{j+1} for $j = 1,2,\dots,p-1$. Then $d(v_1) = 2, d(v_i) = 4\ for\ i = 1,2,\dots,p-1$ and $d(v_p) = 2$. Also, $d(v_k) = 2\ for\ k = p+1, p+2, \dots, 2p-1$. Then, $dd(v_1) = dd(v_p) = 3; dd(v_2) = dd(v_{p-1}) = 7; dd(v_{p+1}) = dd(v_{2p-1}) = 3; dd(v_k) = 6\ for\ k = 3,4,\dots,p-2$ and $dd(v_l) = 4\ for\ l = p+2, p+3, \dots, 2p-2$. Hence for $(p > 2)$,

$$dd\left(R(P_p)\right) = 2\times 3 + 2\times 7 + 2\times 3 + 6(p-4) + 4(p-3) = 10(p-1).$$

Theorem 2.28: Let $R(C_p)$ be a graph of order $2p$, where p is the number of vertices of cycle graph Cp. Then $dd\left(R(C_p)\right) = 10p$.

Proof: Let $v_1, v_2, \dots, v_p$ be the vertices of cycle graph C_p with p edges. Introduce a new vertex corresponding to p edges and then joining each new vertex corresponding edge, we get $R(C_p)$ graph of order $2p$ say $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p}$ and $d_i$is the degree of v_i. Since $v_1, v_2, \dots, v_p$are the vertices of C_p, v_i is adjacent to v_{i-1}, v_{i+1} for $i = 2,3,\dots,p-1$ and v_1 is adjacent to v_p. Choose v_{p+j} is adjacent to v_j, v_{j+1} for $j = 1,2,\dots,p-1$ and v_{2p} is adjacent to v_1, v_p.

Then $d(v_k) = 4\ for\ k = 1,2,\dots,p$ and $d(v_l) = 2\ for\ l = p+1, p+2, \dots, 2p$. Thus, $dd(v_k) = 6\ for\ k = 1,2,\dots,p$ and $dd(v_l) = 4\ for\ l = p+1, p+2, \dots, 2p$. Hence, $dd\left(R(C_p)\right) = 6p + 4p = 10p$.

Theorem 2.29: Let $R(K_{1,p-1})$ be a graph of order $2p-1$, where p is the number of vertices of star graph $K_{1,p-1}(p > 3)$. Then

$$dd\left(R(K_{1,p-1})\right) = 2(p-1)(2p-1).$$

Proof: Let $v_1, v_2, \dots, v_p$ be the vertices of $K_{1,p-1}$ with $p-1$ edges. Introduce a new vertex corresponding to $p-1$ edges and then joining each new vertex to the corresponding edge, we get $R(K_{1,p-1})$ graph of order $2p-1$ say $v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_{2p-1}$ and $d_i$is the degree of v_i. Since $v_1, v_2, \dots, v_p$are the vertices of $K_{1,p-1}$, v_i is adjacent to v_p for $i = 1,2,\dots,p-1$ and v_{p+i} is adjacent to v_i, v_p for $i = 1,2,\dots,p-1$.

Then $d(v_i) = 2 \, for \, i = 1,2, \dots, p-1, d(v_p) = 2(p-1)$ and $d(v_j) = 2 \, for \, j = p+1, p+2, \dots, 2p-1$. Thus, $dd(v_i) = p \, for \, i = 1,2, \dots, p-1$; $dd(v_p) = 2(p-1)^2$ and $dd(v_j) = p \, for \, j = p+1, p+2, \dots, 2p-1$. Hence, $dd\left(R(K_{1,p-1})\right) = 2p(p-1) + 2(p-1)^2 = 2(p-1)(2p-1)$.

Theorem 2.30: $R(K_{1,p-1})$ and $F_3^{(p-1)}$ are equivalent divisor degree of order n = 2p – 1 (p > 3) , where (p – 1) is the number of copies of K3 .

Proof: By theorem 2.17 and 2.29, we have $dd\left(F_3^{(p-1)}\right) = 2(p-1)(2p-1)$ and $dd\left(R(K_{1,p-1})\right) = 2(p-1)(2p-1)$. Hence, $dd\left(F_3^{(p-1)}\right) = dd\left(R(K_{1,p-1})\right)$
and so $R(K_{1,p-1})$ and $F_3^{(p-1)}$ are equivalent divisor degree of order n = 2p – 1.

References:

1. S. Balachandran, Hanyuan Deng, D. Sarala, S. K Ayyaswamy, The Zagreb indices of four operations on graphs, Applied Mathematics and Computation, 275 (2016) 422-431.
2. J.A. Bondy, U.S.R. Murty, Graph Theory with Application, MacMill- lan, New York, 1976.
3. Gutman, The Energy of a Graph, Ber. Math. Stat. Sekt. Forschungszent Graz 103 (1978) 1-22.
4. S.P. Kanniga Devi and K. Nagarajan, Divisor Degree index of Graphs, Int.
5. J. Math. and its Appl. (Accepted).
6. S.P. Kanniga Devi and K. Nagarajan, Divisor Degree Energy of Graphs, Int. J. Math. Archive 8 (2017) 29-35.
7. S. Meenakshi and S. Lavanya, A survey on Energy of Graphs, Annals of Pure and Appl. Math. 8 (2014) 183-191.

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