PROPER LUCKY LABELING OF ROOTED PRODUCT AND CORONA PRODUCT OF CERTAIN GRAPHS

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Abstract: Let $f: V(G) \rightarrow \mathbb{N}$ be a labeling of the vertices of a graph G by positive integers. Let $S(v)$ denote the sum of labels of the neighbors of the vertex $v$in G. If v is an isolated vertex of G we put $S(v) = 0$. A labeling f is lucky if $S(u) \neq S(v)$ for every pair of adjacent vertices $u$and v. The *lucky number* of a graph G, denoted by $\eta(G)$, is the least positive integer k such that G has a lucky labeling with $\{1,2, \dots, k\}$ as the set of labels. A Lucky labeling is proper lucky labeling if the labeling f is proper as well as lucky, i.e. if u and v are adjacent in G then $f(u) \neq f(v)$and $S(u) \neq S(v)$. The proper lucky number of G is denoted by $\eta_p(G)$, is the least positive integer k such that G has a proper lucky labeling with$\{1,2, \dots, k\}$ as the set of labels.

Keywords: Proper lucky labeling, Proper lucky number, Rooted product, Corona product.

Introduction: Lucky labeling is coloring the vertices arbitrarily such that the sum of labels of all adjacent vertices of a vertex is not equal to the sum of labels of all adjacent vertices of any vertex which is adjacent to it. The formal definition was proposed by S. Czerwinski, J. Grytczuk, V. Zelazny[1]. Lucky labeling is applied in real life situations such as transportation network, where pair wise connections are given some numerical values. And each weight could represent the stations or city with certain expenses or costs etc. They are also applicable in computational biology to model protein structures. The relation between chromatic number $\chi(G)$ and clique number $\omega(G)$, is given by $\omega(G) \leq \chi(G)$, where The clique number $\omega(G)$, defined as the size of a largest clique in a graph G [2]. The famous conjecture with regard to lucky number $\eta(G)$ and chromatic number $\leq \chi(G)$ of graph G is worth mentioning here, $\eta(G) \leq \chi(G)$, for every graph G [3].

Proper lucky labeling is a variant of lucky labeling with some more constraints put on the graphs. Proper Lucky labeling is coloring the vertices such that the coloring is proper as well as lucky. Proper lucky labeling was defined by Kins. Yenoke, R.C. Thivyarathi, D. Anthony Xavier [4]. A Lucky labeling is proper lucky labeling if the labeling f is proper as well as lucky, i.e. if u and v are adjacent in G then $f(u) \neq f(v)$and $S(u) \neq S(v)$. The proper lucky number of G is denoted by $\eta_p(G)$, is the least positive integer k such that G has a proper lucky labeling with$\{1,2, \dots, k\}$ as the set of labels.

The graph product is a binary operation on graphs, which takes two graphs G_1 and G_2 and a new graph H is obtained with the following properties: (i) The vertex set of H is the cartesian product $V(G_1) \times V(G_2)$, where $V(G_1)$ and $V(G_2)$ are the vertex set of G_1 and G_2 respectively. (ii) Two vertices of H, (u_1, u_2) and (v_1, v_2) are connected by an edge iff the vertices u_1, u_2, v_1, v_2 satisfy the condition which takes into account the edges of G_1 and G_2. Some of the most common graph products are cartesian graph product, categorical graph product, lexicographic product, strong graph product, co-normal graph product, homomorphic graph product, corona graph product and rooted graph product etc. A rooted graph is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The rooted product graphs were first defined by C. D. Godsil et al. [5]. The corona product graphs were defined by Frucht, R. and Harary, F.[6]. The concept of the corona product has some applications in chemistry for representing chemical compounds [7]. In this paper proper lucky number for rooted product of P_n with P_n, i.e. $P_n o\, P_n$ and corona product of C_n with C_n, i.e. $C_n \odot C_n$ are well computed.

Preliminary definitions and results

This section consists of some basic definitions used in the paper and some results already obtained.

Definition 2.1[1]**:** Let $f: V(G) \to \mathbb{N}$ be a labeling of the vertices of a graph G by positive integers. Let $S(v)$ denote the sum of labels of the neighbors of the vertex $v$in G. If v is an isolated vertex of G we put $S(v) = 0$. A labeling f is lucky if $S(u) \neq S(v)$ for every pair of adjacent vertices $u$and v. The *lucky number* of a graph G, denoted by $\eta(G)$, is the least positive integer k such that G has a lucky labeling with $\{1,2, \dots, k\}$ as the set of labels.

Definition 2.2 [4]: A Lucky labeling is proper lucky labeling if the labeling f is proper as well as lucky, i.e. if u and v are adjacent in G then $f(u) \neq f(v)$and $S(u) \neq S(v)$. The proper lucky number of G is denoted by $\eta_p(G)$, is the least positive integer k such that G has a proper lucky labeling with$\{1,2, \dots, k\}$ as the set of labels.

Definition 2.3 [5]: Given a graph G of order n and a graph H with root vertex v , the rooted product graph $G\, o\, H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the vertex u_i of G with the vertex v in the i^{th} copy of H for every $1 \leq i \leq n$.

Definition 2.4 [6]: The corona product of G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the ith vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \leq i \leq |V(G)|$.

Theorem 2.5[4]: For any connected graph G, $\eta(G) \leq \eta_p(G)$.

Theorem 2.6[4]: For any connected graph G, let ω be its clique number, then $\omega(G) \leq \eta_p(G)$.

Theorem 2.7[4]: Let G be a mesh $M_{n\times n}$. Then the luck number of G is $\eta_p(G) = 2$.

Theorem 2.8[4]: Let G be an extended mesh $EM_{n\times n}$. Then the luck number of G is $\eta_p(G) = 4$.

Theorem 2.9[4]: The proper lucky number of an enhanced mesh $EN_{m\times n}$, $\eta_p(EN_{m\times n}) = 3$.

Theorem 2.10[8]: The proper lucky number of even $TR_{n\times n}$ satisfies $\eta_p(TR_{n\times n}) = 2$.

Theorem 2.11[8]: If n is odd, then the proper lucky number of $TR_{n\times n}$ satisfies $\eta_p(TR_{n\times n}) \leq 6$.

Theorem 3.1: The Rooted product of P_n, P_no P_n admits Proper lucky labeling and $\eta_p(P_n o P_n) = 3$, where ≥ 4.

Proof:

To prove the theorem, the vertices of the product graph are identified as shown in the Fig. 3.1 below.

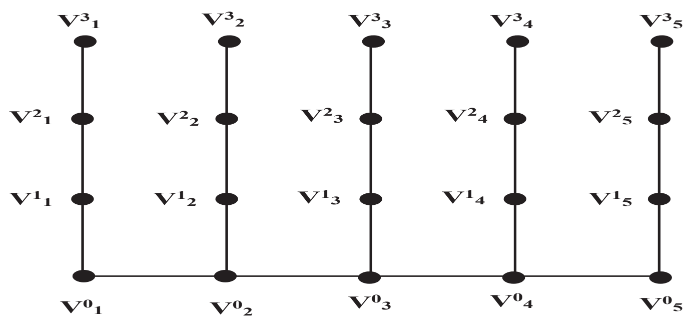

Fig. 3.1 Labeling of P_5 O P_4

For product graph of $P_n \circ P_m$ the number of vertices are mn. In the above graph $n = 5$ and $m = 4$.

Let $f(v_i^j) =$ the label assigned to the vertex v_i^j. Define $S(v_i^j) = \sum_{u_i^j \in N(V_i^j)} f(u_i^j)$, as the sum of neighborhood of the vertex v_i^j, where $N(v_i^j)$ denotes the open neighborhood of $v_i^j \in V$, where $i = 1,2, \dots n$ and $j = 0,1,2 \dots, m$.

There are two cases:

Case1: When m is even.

Label the vertices as follows:

For $j = 1,3,5, \dots, m-2$ and $i = 1,2,3, \dots, n$
$f(V_i^{j-1}) = (3i-1)mod\ 2 + 1.$
For $j = 2,4,6, \dots, (m-2)$ and $i = 1,2,3, \dots, n$
$f(V_i^{j-1}) = (i+2)mod\ 2 + 1.$
For $j = m$ and $i = 1,2,3, \dots, n$
$f(V_i^{j-1}) = (3i-1)mod\ 2 + 1.$
For $j = m-1$ and $i = 1,2,3, \dots, n$
$f(V_i^{j-1}) = \{(i+2)mod\ 2 + 1\} + \{3i\ mod\ 2\}.$

Case2: When m is odd

Label the vertices as follows:
For $j = 1,3,5, \dots, m-2$ and $i = 1,2,3, \dots, n$
$f(V_i^{j-1}) = (3i-1)mod\ 2 + 1.$
For $j = 2,4,6, \dots, (m-2)$ and $i = 1,2,3, \dots, n$.
$f(V_i^{j-1}) = (i+2)mod\ 2 + 1.$
For $j = m$ and $i = 1,2,3, \dots, n$.
$f(V_i^{j-1}) = (3i-1)mod\ 2 + 1.$
For $j = m-1$ and $i = 1,2,3, \dots, n$.
$f(V_i^{j-1}) = \{(3i-1)mod\ 2\} + \{(i+1)mod\ 2 + 1\}.$

In both the cases it is observed that the vertices, labelled as 1 above the row V_i^0 up to V_i^{m-4} row have neighborhood sum as $S(v_i^j) = 4$. In the row V_i^{m-3} the vertices with label 1 have the neighborhood sum as $S(v_i^j) = 5$ and in the row V_i^{m-2} the vertices with label 1 have the neighborhood sum as $S(v_i^j) = 4$. The vertices of the row V_i^{m-1} with label 1 have the neighborhood sum as $S(v_i^j) = 3$. In the row V_i^0 the vertices with label 1 have the neighborhood sum as $S(v_i^j) = 6$ except the corner vertices, i.e. the vertex V_1^0 and the vertex V_n^0 have the neighborhood sum as $S(v_i^j) = 4$.
It is observed that the vertices, labelled as 2 above V_i^0 up to V_i^{m-3} have the neighbourhood sum as $S(v_i^j)$= 2. In the row V_i^{m-1} the vertices with label 2 have the neighborhood sum as $S(v_i^j) = 1$. In the row V_i^0 the vertices with label 2 have the neighborhood sum as $S(v_i^j) = 3$ except the corner vertex, i.e. the vertex V_n^0 has the neighborhood sum as $S(v_i^j) = 2$.
It is observed that the vertices, labelled as 3 have the neighborhood sum as $S(v_i^j) = 2$. Thus, no two adjacent vertices have the same neighborhood sums. Hence the Rooted product of P_n, $P_n \circ P_n$ admits Proper lucky labeling and $\eta_p(P_n \circ P_n) = 3$, when $n \geq 4$. (For illustration see Fig. 3.2)

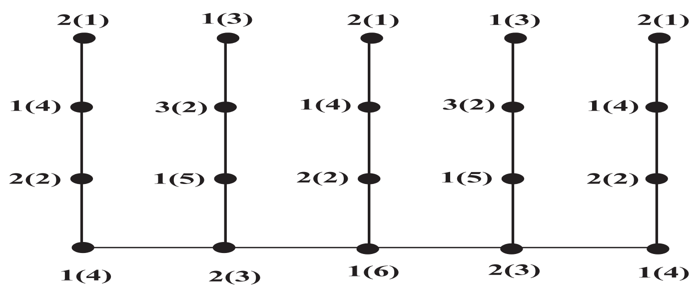

Fig. 3.2 Proper Lucky Labeling of $P_5 \circ P_4$

Hence the Rooted product of P_n, $P_n \circ P_n$ admits Proper lucky labeling and
$\eta_p(P_n \circ P_n) = 3$, where ≥ 4.

Theorem 3.2: The Corona product of C_n, $C_n \odot C_n$ admits Proper lucky labeling and $\eta_p(C_n \odot C_n) = \begin{cases} 4, when\ n\ is\ even \\ 6, when\ n\ is\ odd \end{cases}$, where $n \geq 4$.

Proof: For this theorem two cases arise.

Case 1: When n is even

Label the vertices of the base cycle as 3,4 alternately. The vertices of outer cycle are labeled 1,2 alternately. The neighborhood sum $s(u)$ for the vertices of outer cycles are as follows: $s(u) = 7$ for the vertex with label 1 and adjacent to the vertex with label 3 of the base cycle. $s(u) = 5$ for the vertex with label 2 and adjacent to the vertex with label 3 of the base cycle. $s(u) = 8$ for the vertex with label 1 and adjacent to the vertex with label 4 of the base cycle. $s(u) = 6$ for the vertex with label 2 and adjacent to the vertex with label 4 of the base cycle.

The neighborhood sum $s(u)$ for the base cycle are as follow: The vertex with label 3 has $s(u) = \left\{\frac{n}{2}(3) + 8\right\}$. The vertex with label 4 has $s(u) = \left\{\frac{n}{2}(3) + 6\right\}$. It is observed that no two adjacent vertices have the same $s(u)$ in this case. Hence $\eta_p(C_n \odot C_n) = 4$, when n is even.

Case 2: When n is odd

Label the vertices of the base cycle in clockwise direction alternately with 6,4 and the last vertex is labeled as with 5. Similarly, label the vertices of the outer cycle in clockwise direction with 3,1 alternately and the last vertex is labeled as 2.

The neighborhood sum $s(u)$ for the vertices of outer cycles are as follow: $s(u) = 9\ or\ 10$ for the vertex with label 1 and adjacent to the vertex with label 4 of the base cycle. $s(u) = 8$ for the vertex with label 2 and adjacent to the vertex with label 4 of the base cycle. $s(u) = 6\ or\ 7$ for the vertex with label 3 and adjacent to the vertex with label 4 of the base cycle. $s(u) = 11\ or\ 12$ for the vertex with label 1 and adjacent to the vertex with label 6 of the base cycle. $s(u) = 10$ for the vertex with label 2 and adjacent to the vertex with label 6 of the base cycle. $s(u) = 8\ or\ 9$ for the vertex with label 3 and adjacent to the vertex with label 6 of the base cycle.

The neighborhood sum $s(u)$ for the base cycle are follow: The vertex with label 4 has $s(u) = \left\{\frac{(n-1)}{2}(4) + 14\right\}$, except the vertex, adjacent to the vertex with label 5 in the base cycle, has $s(u) = \left\{\frac{(n-1)}{2}(4) + 13\right\}$. The vertex with label 6 has $s(u) = \left\{\frac{(n-1)}{2}(4) + 10\right\}$. The vertex with label 5 has $s(u) = \left\{\frac{(n-1)}{2}(4) + 12\right\}$. It is observed that no two adjacent vertices have the same $s(u)$ in this case. Hence $\eta_p(C_n \odot C_n) = 6$, when n is odd. (For illustration see the Fig. 3.3)

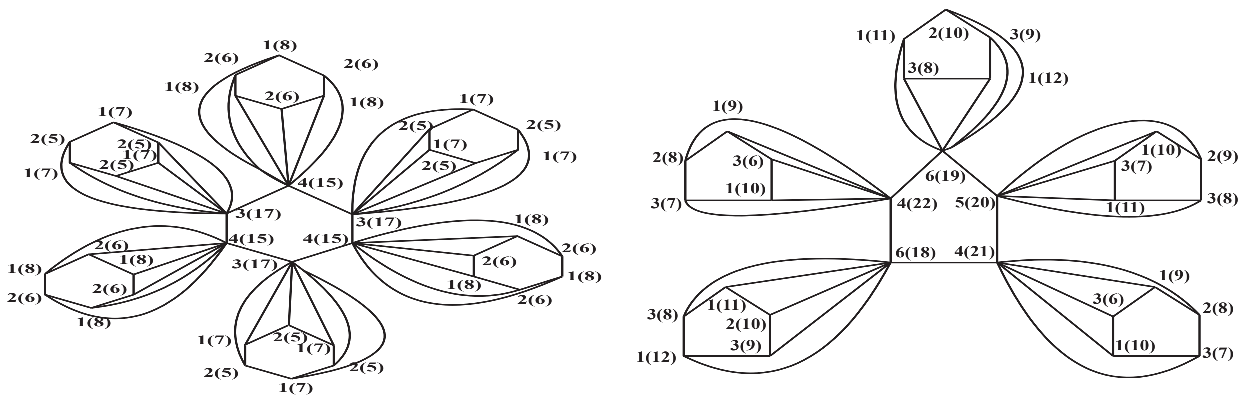

Fig. 3.3 Proper Lucky Labeling of $C_6 \odot C_6$ and $C_5 \odot C_5$

Hence, $\eta_p(C_n \odot C_n) = \begin{cases} 4, when\ n\ is\ even \\ 6, when\ n\ is\ odd \end{cases}$, where $n \geq 4$.

Conclusion: The proper lucky number for rooted product $P_n \circ P_n$ and corona product $C_n \odot C_n$ are well computed.

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