ANALYSIS OF A NON-MARKOVIAN GROUP ARRIVAL QUEUE WITH BALKING AND TWO TYPES OF VARIOUS SERVICES IN CALL CENTERS

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Abstract: The aim of this study is to investigate and analyze some batch arrival queue systems with Bernoulli planned break process and single server providing the service. The study aims to explore and extend the work done on break and unreliable queues with a combination of assumptions like balking and re-service renegation during vacations, time homogeneous random breakdowns and fluctuating modes of service. We investigate the steady state properties, and also the transient behaviour of such queueing systems. Clients can decide not to quit the queue when the server is in either working or vacation (balking). We investigate this phenomenon in the context of a single server with two types of parallel services, Units which join the queue but leave without service in the absence of the server; especially due to vacation is quite a natural phenomenon.

Keywords: Non Markovian, Balking, Reneging, Parallel Service, Random Break.

Introduction: In this paper, we analyze a single server and Bernoulli plans with the server break. The last server is queuing with the arrival of frequent breaks. Just before the start of the service, the customer has the opportunity to choose between different types of services provided. The purpose of this work is, therefore, to generalize the model incorporating resistance and then investigate the behaviour of the system with a new service. The current study is divided into two types of heterogeneous services. We are assuming that we cannot join the system. You cannot reach a unit that arrives when estimating the duration of the wait for a service by the length of the queue. The customer can choose one among the two types of service. We develop the steady-state differential equations for the queuing system by treating the elapsed service time and the elapsed break time as additional variables. We solve these system equations and explicitly derive the probability generation functions of several server states in a random time. A special case is also discussed with the break time of Erlang-k.

We show how we can model telephony in this way and derive standard service level performance measures. We develop a framework to calculate these service levels based on successive service periods. Our approach leads to full characterization of waiting assignment.

Call centers have been a fertile research area for many researchers over the last couple of years [Gans et al., 2003] and [Aksin et al., 2007]. But there are still many important challenges to overcome. One of these answers is how to model customer patience. They are willing to wait for a limited period of time. If they are not served within that time they will give up, ie leave the system. The earliest work mentioned in the literature is found in [Palm, 1937]. In [Stanford, 1979] single-server queues are studied with general service time distributions, tail stability conditions are derived from [Baccelli and Hébuterne, 1981], the distribution of patience depends on the type of call center. show that an Erlang distribution with three phases could work well in some cases. When you focus on impatient customers in a multi-server environment, in [Boxma and de Waal, 1994] insensitive limits and more approximations exist in abandonment probability. [Brandt and Brandt, 1999, Brandt and Brandt, 2002] will consider the state-dependent tail in which arrival rates depend on the number of customers in the system and the number of server tasks. In [Mandelbaum and Zeltyn, 2004], both approximations are based on scaling the queue to obtain estimates for the waiting time distributions. In the context of call centers, [Jouini et al., 2009] study states that there is an impact for announcing delays in a setting of multiple customer classes with Markovian abandonment. Regarding the estimation of the distribution of patience from the actual call center data, published resources are scarce.. However, in [Brown et al., 2005], it was observed that the distribution of patience is not exponential as is usually assumed for call center models in Literature.

In a typical queue system, it is assumed that servers are always available, which is practically unrealistic. However, in reality, servers may not be available for a certain time due to several reasons. The service may be disrupted due to interruptions such as breakdowns in the system. The most important issue with the study of queuing systems involves determining the probability distribution for the duration of the system, the waiting time and the period occupied and a rest period.

A flow of customers from infinite or finite population towards the service facility for receiving some kind of service forms a queue. Queues are experienced in calls waiting in a call center. The temporary absence of the servers for a certain period of time in a queueing system at a service completion instant when there are no customers waiting in a queue or even when there are some is termed as servers break. Arrivals coming and waiting for service can avail the service once the server completes the break period. There are many situations which may lead to a server break for example system maintenance, machine failure, resource sharing, cyclic servers, etc. In the current research, we assume the single break policy following Bernoulli schedule in all the queueing models under study.

Objectives of the research problem :

1. To determine the steady-state behaviour of batch arrival queue systems with two types of a heterogeneous service with a balking. The customer has the option of choosing one of two heterogeneous services.
2. To determine the steady-state behaviour of a batch entry queuing system during the server breaks for Markovian and non-Markovian set up.
3. To determine the time-dependent behaviour and steady state of a single batch arrival server queue with failures and server providing general service in two fluctuating modes.

The model is denoted as $M^X / \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} / 1 / B_x$, a single server with two types of various services.

Notations:

B: Random variable representing completion of break time.

$\mathcal{P}_n(s,t)$: Probability that at time *t*, the server is providing a service, there are *n* (≥ 0) customers in the queue excluding the one customer in service and the elapsed service time of this customer is *s*.

$\mathcal{P}_n(t)$: $\mathcal{P}_n(t) = \int_0^\infty \mathcal{P}_n(s,t)\,ds$, denotes the probability that at time *t*, the server is providing a service, there are *n* (≥ 0) customers in the queue excluding the one being served, irrespective of the value *s*.

$\mathcal{V}_n(s,t)$: Probability that at time *t*, the server is on break with elapsed break time s and there are *n* (≥ 0) customers in the queue waiting for service.

$\mathcal{V}_n(t)$: $\mathcal{V}_n(t) = \int_0^\infty \mathcal{V}_n(s,t)\,ds$, denotes the probability that at time *t*, the server is on break with customers in the queue irrespective of the value *s*.

$\mathcal{P}_{n,i}(s,t)$: Probability that at time t, there are *n* (≥ 0) customers in the queue excluding the one receiving type *i* (*i*=1,2) service and the elapsed service time of this customer is *s*.

$\mathcal{P}_{n,i}(t)$: $\mathcal{P}_{n,i}(t) = \int_0^\infty \mathcal{P}_{n,i}(s,t)\,ds$, denotes the probability that at time *t*, the server is providing a service, there are n (≥ 0) customers in the queue excluding the one being served in type *i* (*i*=1,2) , irrespective of the value *s*.

$\mathcal{Q}(t)$:Probability that at time *t*, there are no customers in the system and the server is idle but available in the system.

Hypotheses describing the model: a) Let $\lambda a_j dt$ be the first order probability that a group of size *j* arrives in a small interval of time $(t, t+dt]$ where $0 \le a_j \le 1$, $\sum_{j=1}^{\infty} a_j = 1$ and $\lambda > 0$ is the mean rate of arrival in groups.

There is a single server providing two types of parallel service to customers, one by one on a first come first served basis (FCFS). Before the service starts, a customer can choose type 1 service with probability p_1 or choose type 2 service with probability p_2 ,where $p_1 + p_2 = 1$. We assume that the service time random variable $X_i,\ i = 1,2$, of type *i* service follows a general probability law with distribution function $\mathcal{G}_i(x)$, probability density function $g_i(x)$ and r^{th} moment $\mathcal{M}(X_i^r)$ $(r = 1,2,3,\ldots)$. Let $\mu_i(x)$ be the conditional probability of completion of type *i* service during the period $(s, s+ds]$

$$\mu_i(x) = \frac{g_i(s)}{1 - \mathcal{G}_i(s)}; i = 1,2,$$

elapsed service time is *s*, so that

$$g_i(x) = \mu_i(x) e^{-\int_0^x \mu_i(s)ds}$$

Once the service of a customer is complete, then the server may decide to take a break with probability p or may continue to serve the next customer with probability or may remain idle in the system if there is no customer requiring service. Further we assume that the break time random variable follows general probability law with distribution function $\mathcal{W}(b)$,the probability density function $w(b)$ and the r^{th} moment $M(\mathcal{B}^r), r = 1,2,\ldots$; Let $f(s)$ be the conditional probability of completion of a break period during the interval $(s, s+ds]$ given that the elapsed vacation time is *s*, so that

$$f(s) = \frac{w(s)}{1 - \mathcal{W}(s)} \text{ and } w(b) = f(b) e^{-\int_0^b f(s)ds}$$

b) Next, we assume that $(1-c_1),(0 \le c_1 \le 1)$ is the probability that an arriving batch balks during the period when the server is busy (available on the system), so that c_1 is the probability of joining the system and $(1-c_2),(0 \le c_2 \le 1)$ is the probability that an arriving batch balks during the period when server is on vacation, so that c_2 joining the system during break period.

The different random process involved in the system is assumed to be independent of each other. The probabilities $\mathcal{P}_{n,i}(t,s), i = 1,2$ for two types of services and $\mathcal{B}_n(t,s)$ for break time and $\mathcal{Q}(t)$ for idle time related for this study.

Now, assuming that the steady state exists, we define the following steady state probabilities:

$$\lim_{t\to\infty} \mathcal{P}_{n,i}(t,s) = \mathcal{P}_{n,i}(s),$$

$$\lim_{t\to\infty} \mathcal{P}_{n,i}(t) = \int_0^\infty \mathcal{P}_{n,i}(t,s)\,ds = \mathcal{P}_{n,i} \tag{1}$$

$$\lim_{t\to\infty} \mathcal{B}_n(t,s) = \mathcal{B}_n(s),$$

$$\lim_{t\to\infty} \mathcal{B}_n(t) = \int_0^\infty \mathcal{B}_n(t,s)\,ds = \mathcal{B}_n, \lim_{t\to\infty} \mathcal{Q}(t) = \mathcal{Q} \tag{2}$$

The subsequent set of differential difference equations represents the queueing system in agreement with assumptions of our call center model

$$\mathcal{P}_{n,i}(t,\Delta s) = (1-\lambda\Delta s)\left(1-\mu_i(s)\Delta s\right)\mathcal{P}_{n,i}(s) +$$

$$\left(1-c_1\right)\left(\lambda\Delta s\right)\mathcal{P}_{n,i}(s) + c_1\lambda\Delta s\sum_{i=1}^{n} a_i\mathcal{P}_{n-j,i}(s) \tag{3}$$

$$, i = 1,2; n \geq 1$$

Connecting the system probabilities at s with $(s+\Delta s)$ by considering $\mathcal{P}_{n,i}(s+\Delta s)$ which means there are $n(n \geq 0)$ customers in the queue excluding one customer in type i (i =1,2) service and the elapsed service time of the customer is $(s, s+\Delta s)$. Then all possible mutually exclusive transitions that can occur during a short interval of time $(s, s+\Delta s)$ are as follows:

(i) There are n customers in the queue excluding one customer in type i (i =1,2) service and elapsed service time is s, no arrivals and no service completion during $(s, s+\Delta s]$. The joint probability for these events is given by $(1-\lambda\Delta s)\left(1-\mu_i(s)\Delta s\right)\mathcal{P}_{n,i}(s)$.

(ii) There are n customers in the queue excluding the one in service since the elapsed service time is s and an arriving batch balks with probability $(1-c_1)$. The joint probability for this case is $\left(1-c_1\right)\left(\lambda\Delta s\right)\mathcal{P}_{n,i}(s)$.

(iii) There are $n-j$ customers in the queue excluding the one in type i (i =1,2) service since the elapsed service time is s, a batch of size j customers arrives and join the queue with probability c_1. The joint probability for this case is: $c_1\lambda\Delta s\sum_{i=1}^{n} a_i\mathcal{P}_{n-j,i}(s)$. From (3)

$$\frac{d}{ds}\left(\mathcal{P}_{n,1}(s)\right) + \left[\lambda + \mu_1(s)\right]\mathcal{P}_{n,1}(s) =$$

$$\lambda(1-c_1)\mathcal{P}_{n,1}(s) + c_1\lambda\sum_{j=1}^{n} a_j\mathcal{P}_{n-j,1}(s); n\geq 1 \tag{4}$$

$$\mathcal{P}_{0,i}(s+\Delta s) = (1-\lambda\Delta s)\left(1-\mu_i(s)\Delta s\right)\mathcal{P}_{0,i}(s)$$

$$+\left(1-c_1\right)(1-\lambda\Delta s)\,\mu_i(s)\mathcal{P}_{1,i}(s), i = 1,2 \tag{5}$$

Connecting the system probabilities at s with $(s+\Delta s)$ by considering $\mathcal{P}_{0,i}(s+\Delta s)$ which means there are no customers in the queue during $(s, s+\Delta s)$. Then all possible mutually exclusive transitions that can occur during a short interval of time $(s, s+\Delta s]$ are as follows:

(i) There are no customers in the queue, no arrivals and no service completion during $(s, s+\Delta s]$. The joint probability for these events is given by $(1-\lambda\Delta s)\left(1-\mu_i(s)\Delta s\right)\mathcal{P}_{0,i}(s)$.

(ii) There is one customer in service of type i, service is completed and the elapsed service time of this customer is s and an arriving batch balks with probability $(1-c_1)$. The joint probability for this case is $\left(1-c_1\right)\left(\lambda\Delta s\right)\mathcal{P}_{1,i}(s)$. From (5)

$$\frac{d}{ds}\left(\mathcal{P}_{0,1}(s)\right)+\left[\lambda+\mu_1(s)\right]\mathcal{P}_{0,1}(s)=$$
$$\lambda(1-c_1)\mathcal{P}_{0,1}(s) \tag{6}$$

$$\frac{d}{ds}\left(\mathcal{P}_{n,2}(s)\right)+\left[\lambda+\mu_2(s)\right]\mathcal{P}_{n,2}(s)=$$
$$\lambda(1-c_1)\mathcal{P}_{n,2}(s)+c_1\lambda\sum_{j=1}^{n}a_j\mathcal{P}_{n-j,2}(s);n\geq 1 \tag{7}$$

$$\frac{d}{ds}\left(\mathcal{P}_{0,2}(s)\right)+\left[\lambda+\mu_2(s)\right]\mathcal{P}_{0,2}(s)=$$
$$\lambda(1-c_1)\mathcal{P}_{0,2}(s) \tag{8}$$

$$\frac{d}{ds}\left(\mathcal{B}_n(s)\right)+\left[\lambda+f(x)\right]\mathcal{B}_n(s)=\lambda(1-c_2)\mathcal{B}_n(s)+c_2\lambda\sum_{j=1}^{n}a_i\mathcal{B}_{n-j}(s);n\geq 1, \tag{9}$$

$$\frac{d}{ds}\left(\mathcal{B}_0(s)\right)+\left[\lambda+f(x)\right]\mathcal{B}_0(s)=$$
$$\lambda(1-c_2)\mathcal{B}_n(s)+c_2\lambda\sum_{j=1}^{n}a_i\mathcal{B}_{n-j}(s);n\geq 1, \tag{10}$$

$$\mathcal{Q}=(1-\lambda)\mathcal{Q}+(1-c_1)\mathcal{Q}+$$
$$(1-p)\left[\int_0^\infty\mathcal{P}_{0,1}(s)\,\mu_1(s)\,ds+\int_0^\infty\mathcal{P}_{0,2}(s)\,\mu_2(s)\,ds\right]$$
$$+\int_0^\infty\mathcal{B}_0(s)\,f(s)\,ds \tag{11}$$

The above equations are to be solved subject to the boundary conditions given below at s = 0

$$\mathcal{P}_{n,1}(0)=(1-p)\,p_1$$
$$\left[\int_0^\infty\mathcal{P}_{n+1,1}(s)\,\mu_1(s)\,ds+\int_0^\infty\mathcal{P}_{n+1,2}(s)\,\mu_2(s)\,ds\right]$$
$$+p_1\int_0^\infty\mathcal{B}_n(s)\,f(s)\,ds+\lambda c_1 p_1 a_{n+1}\mathcal{Q};n\geq 0 \tag{12}$$

$\mathcal{P}_{n,1}(0)$ Probability that at time 0, there are $n(n\geq 0)$ customers in the queue excluding the customer in the service of type1 and elapsed service time of this customer is 0 (service started in type 1). Then we have the following mutually exclusive cases:

(i) There are (n + 1) customers in the queue excluding the one being served in service of type 1 or type 2 given that elapsed service time is s, server completes service of this customer, does not go for vacation and starts serving the next customer in type 1 with probability p_1 .

This is given by the probability $(1-p)\,p_1\left[\int_0^\infty \begin{matrix}\mathcal{P}_{n+1,1}(s)\,\mu_1(s)\,ds+\\ \int_0^\infty\mathcal{P}_{n+1,2}(s)\,\mu_2(s)\,ds\end{matrix}\right]$

(ii) There are (n + 1) customers in the queue and server is on vacation given that the elapsed vacation time is s. Break just completed and starts serving the next customer in type 1 with probability p_1 . This case has the probability $p_1\int_0^\infty \mathcal{B}_n(s)f(s)ds$

(iii) There are no customers in the system, server is idle but available in the system, a batch of size (s + 1) customers arrive and decides to join the queue with probability c_1 . This case has the probability $\lambda c_1 p_1 a_{n+1}\mathcal{Q}$.

$$\mathcal{P}_{n,2}(0)=(1-p)p_2\left[\int_0^\infty \begin{array}{c}\mathcal{P}_{n+1,1}(s)\mu_1(s)ds+\\ \int_0^\infty \mathcal{P}_{n+1,2}(s)\mu_2(s)ds\end{array}\right]+ \qquad (13)$$

$$p_2\int_0^\infty \mathcal{B}_{n+1}(s)f(s)ds+\lambda c_1 p_2 a_{n+1}\mathcal{Q};n\geq 0$$

$$\mathcal{B}_n(0)=p\left[\int_0^\infty \mathcal{P}_{n,1}(s)\mu_1(s)ds+\int_0^\infty \mathcal{P}_{n,2}(s)\mu_2(s)ds\right];n\geq 0 \qquad (14)$$

$\mathcal{B}_n(0)$ = Probability that at time 0, there are $n(n\geq 0)$ customers in the queue and server is on break with elapsed vacation time 0. (break has just started). This has the following mutually exclusive cases:

(i) There are $n(n\geq 0)$ customers in the queue excluding one being served in type 1 or type 2, given that elapsed service time of this customer is s, service is completed and goes for break with probability p. This has the probability

$$p\left[\int_0^\infty \mathcal{P}_{n,1}(s)\mu_1(s)ds+\int_0^\infty \mathcal{P}_{n,2}(s)\mu_2(s)ds\right]$$

(ii) There are (n+1) customers in the queue and server is on break given that the elapsed break time is s.

Break is just completed and starts serving the next customer in type 1 with probability.

This case has the probability $p_1\int_0^\infty \mathcal{B}_{n+1,1}(s)f(s)ds$ and the normalizing condition

$$\mathcal{Q}+\sum_{i=1}^{2}\sum_{n=0}^{\infty}\mathcal{P}_{n,i}(s)ds+\sum_{n=0}^{\infty}\int_0^\infty \mathcal{B}_n(s)ds=1 \qquad (15)$$

Queue size Distribution at Random Epoch

$$\mathcal{P}_i(s,z)=\sum_{n=0}^{\infty}z^n\mathcal{P}_{n,i}(s),\mathcal{P}_i(z)=\sum_{n=0}^{\infty}z^n\mathcal{P}_{n,i},i=1,2 \qquad \mathcal{B}(s,z)=\sum_{n=0}^{\infty}z^n\mathcal{B}_n(s),\mathcal{B}_i(z)=\sum_{n=0}^{\infty}z^n\mathcal{B}_n$$

$$A(z)=\sum_{i=1}^{\infty}z^i a_i,|z|\leq 1 \qquad (16)$$

Now, we multiply equation (4) by z^n , take summations over n from 1 to ∞, adding with (6) and using(15) we get

$$\frac{d}{ds}\left(\mathcal{P}_1(s,z)\right)+\begin{Bmatrix} c_1\left(\lambda-\lambda A(z)\right) \\ +\mu_1(s) \end{Bmatrix}\mathcal{P}_1(s,z)=0 \tag{17}$$

Proceeding similarly for equations(7) – (9), we obtain

$$\frac{d}{ds}\left(\mathcal{P}_2(s,z)\right)+\begin{Bmatrix} c_1\left(\lambda-\lambda A(z)\right) \\ +\mu_2(s) \end{Bmatrix}\mathcal{P}_2(s,z)=0 \tag{18}$$

$$\frac{d}{ds}\left(\mathcal{B}(s,z)\right)+\begin{Bmatrix} c_2\left(\lambda-\lambda A(z)\right)+ \\ f(s) \end{Bmatrix}\mathcal{B}(s,z)=0 \tag{19}$$

We now integrate equations (17) – (19) between limits 0 and s and obtain

$$\mathcal{P}_1(s,z)=\mathcal{P}_1(0,z)e^{-c_1(\lambda-\lambda A(z))s-\int_0^x \mu_1(t)dt};$$

$$\mathcal{P}_2(s,z)=\mathcal{P}_2(0,z)e^{-c_1(\lambda-\lambda A(z))s-\int_0^x \mu_2(t)dt}; \tag{20}$$

$$\mathcal{B}(s,z)=\mathcal{B}(0,z)e^{-c_2(\lambda-\lambda A(z))s-\int_0^x f(t)dt};\lambda>0$$

Further integrating the above equations by parts with respect to s yields

$$\mathcal{P}_1(z)=\mathcal{P}_1(0,z)\frac{\left[1-\mathcal{G}_1^*\left(c_1\left(\lambda-\lambda A(z)\right)\right)\right]}{c_1\left(\lambda-\lambda A(z)\right)};$$

$$\mathcal{P}_2(z)=\mathcal{P}_2(0,z)\frac{\left[1-\mathcal{G}_2^*\left(c_1\left(\lambda-\lambda A(z)\right)\right)\right]}{c_1\left(\lambda-\lambda A(z)\right)} \tag{21}$$

$$\mathcal{B}(z)=\mathcal{B}(0,z)\frac{\left[1-\mathcal{W}^*\left(c_2\left(\lambda-\lambda A(z)\right)\right)\right]}{c_2\left(\lambda-\lambda A(z)\right)}$$

Next we multiply equations (12) with appropriate powers of z, take summation over all possible values of n and using (11) and(16)

$$z\mathcal{P}_1(0,z)=(1-p)p_1 \left[\int_0^\infty \mathcal{P}_1(s,z)\mu_1(s)ds+\int_0^\infty \mathcal{P}_2(s,z)\mu_2(s)ds\right] +p_1\int_0^\infty \mathcal{B}(s,z)f(s)ds+\lambda c_1 p_1(A(z)-1)\mathcal{Q} \tag{22}$$

We perform the similar operations on equations (13) & (14) to obtain

$$z\mathcal{P}_2(0,z)=(1-p)p_2 \left[\int_0^\infty \mathcal{P}_1(s,z)\mu_1(s)ds+\int_0^\infty \mathcal{P}_2(s,z)\mu_2(s)ds\right]+ p_2\int_0^\infty \mathcal{B}(s,z)f(s)ds+\lambda c_1 p_2(A(z)-1)\mathcal{Q} \tag{23}$$

$$\mathcal{B}(0,z) = p \left[\int_0^\infty \mathcal{P}_1(s,z)\mu_1(s)\,ds + \int_0^\infty \mathcal{P}_2(s,z)\mu_2(s)\,ds\right] \tag{24}$$

Now multiplying equations (20) by $\mu_1(s), \mu_2(s)$ respectively then integrating the resulting equations over s, we get $\int_0^\infty \mathcal{P}_1(s,z)\mu_1(s)\,ds = \mathcal{P}_1(0,z)\mathcal{G}_1^*\left(c_1\left(\lambda - \lambda A(z)\right)\right)$ (25)

$$\int_0^\infty \mathcal{P}_2(s,z)\mu_2(s)\,ds = \mathcal{P}_2(0,z)\mathcal{G}_2^*\left(c_1\left(\lambda - \lambda A(z)\right)\right) \tag{26}$$

Similarly multiplying equation (20) by $f(s)$, we get

$$\int_0^\infty \mathcal{B}(s,z) f(s)\,ds = \mathcal{B}(0,z)\mathcal{W}^*\left(c_2\left(\lambda - \lambda A(z)\right)\right) \tag{27}$$

Where $\mathcal{G}_1^*\left(c_1\left(\lambda - \lambda A(z)\right)\right) = \int_0^\infty e^{-c_1(\lambda-\lambda A(z))s}\,d\mathcal{G}_i(s)$ is the Laplace-Stieltjes transform of type i, (i=1,2) service and $\mathcal{W}^*\left(c_2\left(\lambda - \lambda A(z)\right)\right) = \int_0^\infty e^{-c_2(\lambda-\lambda A(z))s}\,d\mathcal{W}(s)$ is the Laplace-Stieltjes transform of break time.

Substituting relation (25) – (27) in equations (22) and (24)
and denoting $c_1\left(\lambda - \lambda A(z)\right) = m, c_2\left(\lambda - \lambda A(z)\right) = n$,
we get

$$z\mathcal{P}_1(0,z) = (1-p)\,p_1 \begin{bmatrix} \mathcal{P}_1(0,z)\mathcal{G}_1^*(m) \\ +\mathcal{P}_2(0,z)\mathcal{G}_2^*(m) \end{bmatrix} + p_1\mathcal{B}(0,z)\mathcal{W}^*(n) + \lambda c_1 p_1 \left(A(z)-1\right)\mathcal{Q} \tag{28}$$

$$z\mathcal{P}_2(0,z) = (1-p)\,p_2 \begin{bmatrix} \mathcal{P}_1(0,z)\mathcal{G}_1^*(m) + \\ \mathcal{P}_2(0,z)\mathcal{G}_2^*(m) \end{bmatrix} + p_1\mathcal{B}(0,z)\mathcal{W}^*(n) + \lambda c_1 p_2 \left(A(z)-1\right)\mathcal{Q} \tag{29}$$

$$\mathcal{B}(0,z) = p\left[\mathcal{P}_1(0,z)\mathcal{G}_1^*(m) + \mathcal{P}_2(0,z)\mathcal{G}_2^*(m)\right] \tag{30}$$

Utilizing (25) –(27) in equations (22) and (24) we have

$$z\mathcal{P}_1(0,z) = (1-p)\,p_1 \begin{bmatrix} \mathcal{P}_1(0,z)\mathcal{G}_1^*(m) + \\ \mathcal{P}_2(0,z)\mathcal{G}_2^*(m) \end{bmatrix} + p_1 p \begin{bmatrix} \mathcal{P}_1(0,z)\mathcal{G}_1^*(m) + \\ \mathcal{P}_2(0,z)\mathcal{G}_2^*(m) \end{bmatrix} \mathcal{W}^*(n) + \lambda c_1 p_1 \left(A(z)-1\right)\mathcal{Q} \tag{31}$$

Similarly

$$z\mathcal{P}_2(0,z) = (1-p)\,p_1\begin{bmatrix}\mathcal{P}_1(0,z)\,\mathcal{G}_1^*(m) + \\ \mathcal{P}_2(0,z)\,\mathcal{G}_2^*(m)\end{bmatrix} +$$

$$p_2 p\begin{bmatrix}\mathcal{P}_1(0,z)\,\mathcal{G}_1^*(m) + \\ \mathcal{P}_2(0,z)\,\mathcal{G}_2^*(m)\end{bmatrix}\mathcal{W}^*(n) \quad (32)$$

$$+\lambda c_1 p_2\left(A(z)-1\right)\mathcal{Q}$$

Solving equations (31) and (32) we obtain

$$\mathcal{P}_1(z) = p_1\frac{\left[1-\mathcal{G}_1^*(m)\right]\mathcal{Q}}{z^2 - z\begin{bmatrix}(1-p)+ \\ p\mathcal{W}^*(n)\end{bmatrix}\begin{bmatrix}p_1\mathcal{G}_1^*(m)+ \\ p_2\mathcal{G}_2^*(m)\end{bmatrix}}$$

$$\mathcal{P}_2(z) = p_2\frac{\left[1-\mathcal{G}_2^*(m)\right]\mathcal{Q}}{\begin{bmatrix}(1-p)+ \\ p\mathcal{W}^*(n)\end{bmatrix}\begin{bmatrix}p_1\mathcal{G}_1^*(m)+ \\ p_2\mathcal{G}_2^*(m)\end{bmatrix}} \quad (33)$$

$$\mathcal{B}(z) = \frac{p\dfrac{c_1}{c_2}\left[1-\mathcal{W}^*(n)\right]\begin{bmatrix}p_1\mathcal{G}_1^*(m)+ \\ p_2\mathcal{G}_2^*(m)\end{bmatrix}\mathcal{Q}}{\left[(1-p)+p\mathcal{W}^*(n)\right]\left[p_1\mathcal{G}_1^*(m)+p_2\mathcal{G}_2^*(m)\right]}$$

Let us now define $\mathcal{P}_q(z)$ as the probability generating function of the queue size irrespective of the type of service the server is providing, such that adding equations, we get

$$\mathcal{P}_q(z) = \mathcal{P}_1(z) + \mathcal{P}_2(z) + \mathcal{B}(z) \quad (34)$$

Now we have $$\mathcal{Q} = 1 - \frac{c_1\lambda E(I)\begin{bmatrix}p_1 E(X_1)+ \\ p_2 E(X_2) + pE(\mathcal{B})\end{bmatrix}}{1+p(c_1-c_2)\lambda E(I)E(\mathcal{B})} \quad (35)$$

Where $E(I)$ is the mean size of batch arriving customers, $E(X_1), E(X_2)$ are the mean service times of type 1 and type 2 services respectively. $E(\mathcal{B})$ is the mean of break time

And the utilization factor $\rho = 1-\mathcal{Q}$ is given by

$$\rho = \frac{c_1\lambda E(I)\left[p_1 E(X_1)+p_2 E(X_2)\right]}{1+p(c_1-c_2)\lambda E(I)E(\mathcal{B})} < 1 \quad (36)$$

The probability generating function of the queue size irrespective of system for a $M^X/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1/B_x$ queue with balking during busy and break periods.

$$\mathcal{P}_q(z) = \frac{\mathcal{Q}\left[\left[\begin{array}{l} p_1\mathcal{G}_1^*(m) + \\ p_2\mathcal{G}_2^*(m) - 1 \end{array}\right]\right] + p\frac{c_1}{c_2}\left[\begin{array}{l} \mathcal{W}^*(n) \\ -1 \end{array}\right] \left[\begin{array}{l} p_1\mathcal{G}_1^*(m) \\ +p_2\mathcal{G}_2^*(m) \end{array}\right]}{z - \left[\begin{array}{l} (1-p) \\ +p\mathcal{W}^*(n) \end{array}\right]\left[\begin{array}{l} p_1\mathcal{G}_1^*(m) \\ +p_2\mathcal{G}_2^*(m) \end{array}\right]} \quad (37)$$

Conclusion: We have introduced a reversal on a single server queue containing two types of parallel services and two service steps that are classified as two different models. Arrivals are assumed to be in batches following a Poisson process. A new model is developed by generalizing the model with two service steps considering optional re-service. To the best of our knowledge, no study in the literature has combined the two hypotheses of retraction and re-service in an MX /G / 1 queuing system with server pauses with variants of service, which led to the motivation behind this study.

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