FIXED POINT THEOREMS IN FUZZY METRIC SPACES USING PROPERTY JCLRts

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Abstract: In this paper we use the concept of JCLRts property which enables us to solve problem in a easy and lucid way .This property was used by sintunavarat in their paper.We also used the concept of compatible mapping of type β which enhances our problem solving strategy.

Keywords: Common limit in the range of g property, Property (E.A.), Joint common limit in the range of T and S property, Compatible of type (β), Fuzzy metric space, Weakly compatible , JCLRts property.

Introduction and Preliminary Concepts: In 1975, Kramosil and Michalek [24] introduced the notion of fuzzy metric space which could be considered as generalization of probabilistic metric space due to Menger [25]. George and Veeramani [19, 20], modified the notion given by Kramosil and Michalek, in order to introduce a Hausdorff topology on fuzzy metric spaces. Many authors have contributed to the development of this theory and its applications, for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 32, 15, 17, 18]. In 2002, Aamri and El Moutawakil [5] defined the property (E.A.) for self-mappings whose class contains the class of non compatible as well as compatible mappings. It is observed that the property (E.A.) requires the containment and closedness of ranges for the existence of fixed points. Later on, Sintunavarat and Kumam [29] coined the idea of "common limit in the range property" which does not require the closedness of the subspaces for the existence of fixed point for a pair of mappings. Recently, Chauhan et al. [18] defined the notion of $JCLR$ property which does not require closeness of subspaces for the existence of fixed points for two pairs of mappings.

Definition 1.1[29]: A pair (A, B) of self mappings of a fuzzy metric space $(X, M,*)$ is said to satisfy the "common limit in the range of g" property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Bu$$

Definition 1.2 [30]: Let $(X, M,*)$ be a fuzzy metric space and $A, B, S, T : X \to X$.The pair $(A, T)$and(B, S) are said to satisfy the "Joint common limit in the range of T and S" property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = Tu = Su.$$

Definition 1.2 [20] : Two self mappings A and B of a non empty set X are said to be weakly compatible if they commute at their coincidence point, i.e. if

$$Az = Bz$$

for some z $\in X$, then
$ABz = BAz$.
Remark 1.1: If $T = S$ and $A = B$ in (6.1.2) then we get definition of (CLR_g)

Remark 1.2 [21]: In a fuzzy metric space $(X, M,*)$, $M(x, y, .)$ is non- decreasing for all $x, y \in X$.

Remark 1.3:[22]: If the pair (A, S) is compatible of type (β) or compatible of type (α) , then it is weak compatible.

Remark 1.4 [28]: If the pair (A, S) is semi compatible, then it is weak compatible

Lemma 1.1 [26]: Let $(X, M, *)$ be a fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0,1)$ $M(x, y, kt) \geq M(x, y, t)$then $x = y$.

Example 1.1 Let $(X, M, *)$ be a fuzzy metric space with X = [0,∞) and

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|} & if\ t > 0 \\ 0 & if\ \ t = 0 \end{cases}$$

for all $x, y \in X$.
Define self mappings A and B on X by
$A(x) = 2x + 5, B(x) = 7x$
Let a sequence

$$\{x_n\} = \left\{1 + \frac{1}{n}\right\}_{n \in N}$$

in X we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 7 = B(1) \in X$$

which shows that A and B satisfy the (CLR_g) property.

Example 1.2: Let$(X, M, *)$ be a fuzzy metric space, where X = [0, 6] with t-norm defined $a * b = \min\{a, b\}$, for all $a, b \in [0,1]$ and

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

for all $t > 0$ and $x, y \in X$. Define self mappings A and S on X as follows,

$Sx = \begin{cases} 6 & \text{if } 0 \leq x \leq 1 \\ \frac{x}{6} & 1 < x \leq 6 \end{cases}$ $Tx = \begin{cases} 6 & x = 1 \\ \frac{x+6}{12} & \text{otherwise} \end{cases}$

Then we have
$S(1) = T(1) = 6$ and $S(6) = T(6) = 1$
Also
$ST(1) = TS(1) = 1$ and $ST(6) = TS(6) = 6$
Thus (A, S) is weak compatible.

Theorem 1.1[23]: Let(X, d) be a cone metric space, P be a normal cone with normal constant K, q be a c-distance on X and S,T: $X \to$ X such that $T(X) \subseteq S(X)$ and $S(X)$ be a complete subspace of X. Suppose that there exists mappings $\alpha, \beta, \gamma, \mu: X \to [0,1)$ such that the following assertion hold;
(1.1.1) $(Tx) \leq \alpha(Sx)$, $\beta(Tx) \leq \beta(Sx)$, $\gamma(Tx) \leq \gamma(Sx)$ and $\mu(Tx) \leq \mu(Sx)$ for all $x \in X$,
(1.1.2)$(\alpha + 2\beta + \gamma + \mu)(x) < 1$ for all $x \in X$,
(1.1.3) $q(Tx, Ty) \leq \alpha(Sx)q(Sx, Sy) + \beta(Sx)q(Sx, Ty) + \gamma(Sx)q(Sx, Tx) + \mu(Sx)\, q(Sy, Ty)$
for all $x, y \in X$,
(1.1.4) $\text{Inf}\{\|q(Tx, y)\| + \|q(Sx, y)\| + \|q(Sx, Tx)\|: x \in X\} > 0$ for all $x, y \in X$ with $Tx \neq y$ or Sy ≠ y then S and T have a common fixed point in X.
We prove the following theorem using property (JCLRts) and compatibility of type (β).

Theorem 1.2 : Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be mappings from X into itself, $(\alpha + \beta + \gamma) = 1$ and there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$;
(1.2.1) The pairs (A, T) and $(B, S)$are compatible of type (β),
(1.2.2) $M(Ax, By, kt) \geq \frac{2\alpha(M(Ax,Tx,t)*M(By,Tx,t))}{1+max\{M(Ax,By,t),M(Ax,Tx,t),M(By,Tx,t)\}}$
$+\beta(M(Ax, Sy, t) * M(By, Tx, t)) + \gamma(M(Ax, By, t) * M(Ax, Tx, t))$,
(1.2.3)(A, T) and (B, S) satisfy $(JCLRts)$ property, then
A, B, S and T have a common fixed point.
Proof:Since the pairs (A, T) and (B, S) satisfy $(JCLRts)$ property , there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such

that$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = Su = Tu$ for some$u \in X$.
Now we assert that $Bu = Su$
Putting $x = x_n$and$y = u$ in inequality(1.2.2)

$$M(Ax_n, Bu, kt) \geq \frac{2\alpha(M(Ax_n, Tx_n, t) * M(Bu, Tu, t))}{1 + max\{M(Ax_n, Tx_n, t), M(Ax_n, Bu, t), M(Bu, Tx_n, t)\}}$$

$+\beta(M(Ax_n, Su, t) * M(Bu, Tx_n, t)) + \gamma(M(Ax_n, Bu, t) * M(Ax_n, Tx_n, t))$

$$M(Su, Bu, kt) \geq \frac{2\alpha\big(M(Su, Su, t) * M(Bu, Su, t)\big)}{1 + max\{M(Su, Su, t), M(Bu, Su, t), M(Bu, Su, t)\}}$$

$+\beta(M(Su, Su, t) * M(Bu, Su, t)) + \gamma(M(Su, Bu, t) * M(Su, Su, t))$

$M(Su, Bu, kt) \geq \frac{2\alpha\left(1*M(Bu,Su,t)\right)}{1+max\,\{1,M(Bu,Su,t),M(Bu,Su,t)\}} +\beta\big(1 * M(Bu, Su, t)\big)$
$+\gamma\big(1 * M(Bu, Su, t)\big)$
$M(Su, Bu, kt) \geq \alpha M(Bu, Su, t)+\beta M(Bu, Su, t) + \gamma M(Bu, Su, t)$

$$M(Su, Bu, kt) \geq (\alpha + \beta + \gamma)M(Bu, Su, t)$$

$$M(Su, Bu, kt) \geq M(Bu, Su, t)$$

Therefore by using lemma (1.1) we have

$$Su = Bu$$

Now we show that
Au = Bu
Putting $x = u$and$y = y_n$ in inequality (1.2.2) we have

$$M(Au, By_n, kt) \geq \frac{2\alpha(M(Au, Tu, t) * M(By_n, Tu, t))}{1 + max\{M(Au, By_n, t), (Au, Tu, t), M(By_n, Tu, t)\}}$$

$+\beta(M(Au, Sy_n, t) * M(By_n, Tu, t)) + \gamma(M(Au, By_n, t) * M(Au, Tu, t))$

$$M(Au, Bu, kt) \geq \frac{2\alpha(M(Au, Tu, t) * M(Su, Su, t))}{1 + max\{M(Au, By_n, t), M(Au, Bu, t), M(By_n, Tu, t)\}}$$

$+\beta(M(Au, Bu, t) * M(Su, Su, t)) + \gamma(M(Au, Bu, t) * M(Au, Bu, t))$

$$M(Au, Bu, kt) \geq (\alpha + \beta + \gamma)M(Au, Bu, t)$$

$$M(Au, Bu, kt) \geq M(Au, Bu, t)$$

Therefore by using lemma (1.1) we have

$$Au = Bu$$

Now we assume that $z = Au = Bu = Su = Tu$
Since the pair (A, T) is compatible of type (β), therefore it is also weakly compatible by remark (1.3) , hence $ATu = TAu$.

Then $Az = ATu = TAu = Tz$.

Now (B, S) is compatible of type (β), therefore it is also weakly compatible by remark (1.3) , therefore $Bz = BSu = SBu = Sz$
We show that

$$z = Az$$

Putting $x = z$ and y = u in inequality (1.2.2)

$$M(Az, Bu, kt) \geq \frac{2\alpha(M(Az, Tz, t) * M(Bu, Tz, t))}{1 + max\{M(Az, Bu, t), M(Az, Tz, t), M(Bu, Tz, t)\}}$$

$+\beta(M(Az, Su, t) * M(Bu, Tz, t)) + \gamma(M(Az, Bu, t) * M(Az, Tz, t))$

$$M(Az, z, kt) \geq \frac{2\alpha(M(Az, z, t) * M(z, Az, t))}{1 + max\{M(Az, z, t), M(Az, Az, t), M(z, Az, t)\}}$$

$+\beta(M(Az, z, t) * M(z, Az, t)) + \gamma(M(Az, z, t) * M(Az, Az, t))$

$$M(Az,z,kt) \geq \frac{2\alpha(1 * M(z,Az,t))}{1 + max\{M(Az,z,t),1,M(z,Az,t)\}}$$
$+\beta\ (M(Az,z,t) * M(z,Az,t)) + \gamma(M(Az,z,t) * 1)$
$$M(Az,z,kt) \geq (\alpha + \beta + \gamma)M(z,Az,t)$$
$$M(Az,z,kt) \geq M(z,Az,t)$$

Therefore by using lemma (1.1) we have
$$Az = z$$
Hence
$$Az = Tz$$

Next we show that
$$z = Bz$$
Putting $x = u$ and $y = z$ in inequality (1.2.2)

$$M(Au,Bz,kt) \geq \frac{2\alpha(M(Au,Tu,t) * M(Bz,Tu,t))}{1 + max\{M(Au,By,t),M(Au,Tu,t),M(By,Tu,t)\}}$$
$+\beta(\ M(Au,Sz,t) * M(Bz,Tu,t)) + \gamma(M(Au,Bz,t) * M(Au,Tu,t))$

$M(z,Bz,kt) \geq \frac{2\alpha(M(z,z,t)*M(Bz,Tu,t))}{1+max\,\{M(z,Bz,t),M(z,z,t),M(Bz,z,t)\}}+\beta(M(z,Bz,t) * M(Bz,z,t)) + \gamma(M(z,Bz,t) * M(z,z,t))$

$M(z,Bz,kt) \geq \frac{2\alpha(1*M(Bz,Tu,t))}{1+max\,\{M(z,Bz,t),1,M(Bz,Tu,t)\}}+ \beta\big(M(z,Bz,t) * M(Bz,z,t)\big)$

$+\gamma(M(z,Bz,t) * 1)$

$$M(z,Bz,kt) \geq (\alpha + \beta + \gamma)M(z,Bz,t)$$
$$M(z,Bz,kt) \geq M(z,Bz,t)$$
Therefore by using lemma (1.1) we have
$$Bz = z$$
Hence
$z = Bz = Sz$,therefore $z = Bz = Sz = Az = Tz$ implies that A,B,S and T have a common fixed point.

Uniqueness: Let w be another fixed point of the mappings A,B,S and T.
Putting $x = z$ and $y = w$ in inequality (1.2.2) we have

$M(Az,Bw,kt) \geq \frac{2\alpha(M(Az,Tz,t) * M(Bw,Tz,t))}{1 + max\{M(Az,Bw,t),M(Az,Tz,t),M(Bw,Tz,t)\}}$

$+\ \beta(\ M(Az,Sw,t) * M(Bw,Tz,t)) + \gamma(M(Az,Bw,t) * M(Az,Tz,t))$

$M(z,w,kt) \geq \frac{2\alpha(M(z,z,t)*M(w,z,t))}{1+max\,\{M(z,w,t),M(z,z,t),M(w,z,t)\}}+\beta\big(M(z,w,t) * M(w,z,t)\big) + \gamma(M(z,w,t) * M(z,z,t))$

$M(z,w,kt) \geq \frac{2\alpha(1*M(w,z,t))}{1+max\,\{M(z,w,t),1,M(w,z,t)\}}+\beta(M(z,w,t) * M(w,z,t))$

$+\gamma(M(z,w,t) * 1)$
$$M(z,w,kt) \geq (\alpha + \beta + \gamma)M(z,w,t)$$
$$M(z,w,kt) \geq M(z,w,t)$$
Therefore by using lemma (1.1) we have
$z = w$.
This completes the proof.

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