STRONG FORM OF GENERALIZED CLOSED SETS IN N-TOPOLOGY

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Abstract: The motive of this research topic is to introduce and evolve another innovative class of sets namely $N\tau\theta$-closed sets defined in terms of $N\tau\theta$-closure which are stronger than the class of $N\tau$-open sets. Further the idea of $N\tau\theta$-open sets is extended to induce and investigate $N\tau\theta$-interior.
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Introduction: Topology is a major area of mathematics concerned with properties that are preserved under continuous deformations of objects such as deformations that involve stretching, but not tearing or gluing. In 1963, J.C. Kelly [2] established the important concept of bitopological space ie, a non-empty set X equipped with two arbitrary topologies τ_1, τ_2. Lellis Thivagar et al.[3] went beyond imagination by introducing and establishing the concept of N-topological space, namely a non-empty set X together with N arbitrary topologies. He also investigated the general formula to determine the N-topological open sets. The concept of θ-open, θ-closed, θ-interior and θ-closure were introduced by Velicko [5] to study the class of H-closed spaces. Later, Noiri [4] and Jafari [1] obtained several new and interesting results related to these sets.

In this article we want to introduce and develop yet another innovative class of sets namely $N\tau\theta$-closed sets defined in terms of $N\tau\theta$-closure which are stronger than the class of $N\tau$-open sets. Also we introduce the notion of $N\tau\theta$-open sets and investigate $N\tau\theta$-interior.

Preliminaries: In this section we discuss some basic properties of N-topological spaces that are useful in the sequel. By a space $(X, N\tau)$, we mean a N-topological space on X in which no separation axioms are assumed unless explicitly stated.

Definition 2.1 [3] Let X be a non empty set and τ_1, τ_2, ..., τ_N be N-arbitrary topologies defined on X. The collection $N\tau = \{S \subseteq X : S = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i),\ A_i, B_i \in \tau_i\}$ is called a N-topology on X if the following axioms are satisfied:

1. $X, \varnothing \in N\tau$.
2. $\bigcup_{i=1}^{\infty} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{\infty} \in N\tau$.
3. $\bigcap_{i=1}^{n} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{n} \in N\tau$.

Then $(X, N\tau)$ is called a N-topological space on X. The elements of $N\tau$ are known as $N\tau$-open sets on X and its complement is called as $N\tau$-closed on X. We denote $N\tau O(X,x)$ as the set of all $N\tau$-open sets containing x on X. The set of all $N\tau$-open sets and the set of all $N\tau$-closed sets on X are denoted by $N\tau$ O(X) and $N\tau$ C(X) respectively.

Definition 2.2 [3] The interior and closure of a subset A of $(X, N\tau)$ are respectively defined as

1. $N\tau\text{-}int(A) = \cup\{G : G \subseteq A$ and G is $N\tau$-open$\}$.
2. $N\tau\text{-}cl(A) = \cap\{F : A \subseteq F$ and F is $N\tau$-closed$\}$.

Theorem 2.3 [3] Let $(X, N\tau)$ be a N-topological space on X and let $A, B \subseteq X$. Then

1. $N\tau\text{-}cl(A)$ is the smallest $N\tau$-closed set containing A.
2. A is $N\tau$-closed if and only if $N\tau\text{-}cl(A) = A$. In particular, $N\tau\text{-}cl(\varnothing) = \varnothing$ and $N\tau\text{-}cl(X) = X$.
3. $A \subseteq B \Rightarrow N\tau\text{-}cl(A) \subseteq N\tau\text{-}cl(B)$.
4. $N\tau\text{-}cl(A \cup B) = N\tau\text{-}cl(A) \cup N\tau\text{-}cl(B)$
5. $N\tau\text{-}cl(A \cap B) \subseteq N\tau\text{-}cl(A) \cap N\tau\text{-}cl(B)$.
6. $N\tau\text{-}cl(N\tau\text{-}cl(A)) = N\tau\text{-}cl(A)$.

Theorem 2.4 [3] Let $(X, N\tau)$ be a N-topological space on X and $A \subseteq X$. Then $x \in N\tau\text{-}cl(A)$ if and only if $G \cap A \neq \varnothing$ for every $N\tau$-open set G containing x.

Theorem 2.5 [3] Let $(X, N\tau)$ be a N-topological space on X and $A \subseteq X$. Then

1. $N\tau\text{-}int(X - A) = X - N\tau\text{-}cl(A)$.
2. $N\tau\text{-}int(A) \supseteq \tau_1\text{-}int(A) \cup \tau_2\text{-}int(A) \cup \ldots \cup \tau_N\text{-}int(A)$.
3. $N\tau\text{-}cl(X - A) = X - N\tau\text{-}int(A)$.
4. $N\tau\text{-}cl(A) \subseteq \tau_1\text{-}cl(A) \cap \tau_2\text{-}cl(A) \cap \ldots \cap \tau_N\text{-}cl(A)$.

θ-Closed Sets in N-Topological Spaces: ***In this section we introduce θ-open and θ-closed sets in N-topological space and establish their relationships with suitable examples***.

Definition 3.1 Let $(X, N\tau)$ be a N-topological space on X and $A \subseteq X$. An element $x \in X$ is said to be $N\tau\text{-}\theta$ cluster point of A if $A \cap N\tau\text{-}cl(G) \neq \varnothing$ for every $N\tau$-open set G containing x. The set of all $N\tau\text{-}\theta$-cluster points of A is called $N\tau\text{-}\theta$ closure of A and is denoted by $N\tau\text{-}cl_\theta(A)$. A subset A of X is said to be $N\tau\theta$-closed in X if $N\tau\text{-}cl_\theta(A) = A$ and its complement is called $N\tau\theta$-open.

Example 3.2 Let $N = 2$, $X = \{a,b,c\}$, consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$ and $\tau_2 O(X) = \{\varnothing, X, \{b,c\}\}$. Then $2\tau O(X) = \{\varnothing, X, \{a\}, \{b,c\}\}$ is a bitopology and if $A = \{b,c\}$, then $2\tau\text{-}cl_\theta(A) = \{b,c\} = A$. Hence A is $2\tau\theta$-closed and the complement set $\{a\}$ is $2\tau\theta$-open.

Theorem 3.3 $A \subseteq N\tau\text{-}cl_\theta(A)$, for any subset A of $(X, N\tau)$.

Proof: If $x \in A$ and G is a $N\tau$-open set containing x, then $G \subseteq N\tau\text{-}cl(G)$ and hence $x \in N\tau\text{-}cl(G)$. Thus $x \in A \cap N\tau\text{-}cl(G)$ and therefore, $A \cap N\tau\text{-}cl(G) \neq \varnothing$ for every $N\tau$-open set G containing x. Hence $x \in N\tau\text{-}cl_\theta(G)$. That is $A \subseteq N\tau\text{-}cl_\theta(A)$.

Theorem 3.4 $N\tau\text{-}cl(A) \subseteq N\tau\text{-}cl_\theta(A)$, for any subset A of $(X, N\tau)$.

Proof: If $x \in N\tau\text{-}cl(A)$, then $G \cap A \neq \varnothing$ for every $N\tau$-open set G containing x. Since $G \subseteq N\tau\text{-}cl(G)$, $G \cap A \subseteq N\tau\text{-}cl(G) \cap A$ and hence $N\tau\text{-}cl(G) \cap A \neq \varnothing$. Therefore $x \in N\tau\text{-}cl_\theta(A)$. Thus, $N\tau\text{-}cl(A) \subseteq N\tau\text{-}cl_\theta(A)$.

Remark 3.5 $N\tau\text{-}cl(A) \neq N\tau\text{-}cl_\theta(A)$. For example, let $N = 3$, $X = \{a,b,c\}$, consider $\tau_1O(X) = \{\varnothing, X, \{a\}\}$, $\tau_2O(X) = \{\varnothing, X, \{a,c\}\}$, $\tau_3O(X) = \{\varnothing, X\}$. Then $3\tau O(X) = \{\varnothing, X, \{a\}, \{a,c\}\}$ is a tritopology and if $A = \{c\}$, then $3\tau\text{-}cl(A) = \{b,c\}$ and $3\tau\text{-}cl_\theta(A) = X$. Hence $3\tau\text{-}cl(A) \neq 3\tau\text{-}cl_\theta(A)$.

Theorem 3.6 If A is $N\tau$-open set in a N-topological space $(X, N\tau)$, then $N\tau\text{-}cl(A) = N\tau\text{-}cl_\theta(A)$

Proof: Let A be $N\tau$-open set in X. We know that $N\tau\text{-}cl(A) \subseteq N\tau\text{-}cl_\theta(A)$. Let $x \in N\tau\text{-}cl_\theta(A)$, then $A \cap N\tau\text{-}cl(G) \neq \varnothing$ for every $N\tau$-open set G containing x. If $A \cap G = \varnothing$, $G \subseteq A^c$. Since A is $N\tau$-open, A^c is $N\tau$-closed. That is A^c is $N\tau$-closed set containing G. But $N\tau\text{-}cl(G)$ is the smallest $N\tau$-closed set containing G. Therefore, $N\tau\text{-}cl(G) \subseteq A^c$. Thus, $A \cap N\tau\text{-}cl(G) = \varnothing$ which is a contradiction. Therefore $A \cap G \neq \varnothing$ for every $N\tau$-open set G containing x. That is $x \in N\tau\text{-}cl(A)$. Therefore, $N\tau\text{-}cl(A) \supseteq N\tau\text{-}cl_\theta(A)$. Hence $N\tau\text{-}cl(A) = N\tau\text{-}cl_\theta(A)$.

Theorem 3.7 $N\tau\text{-}cl_\theta(A) \subseteq N\tau\text{-}cl_\theta(N\tau\text{-}cl_\theta(A))$, for any subset A of $(X, N\tau)$.

Proof: Proof follows from theorem 3.3 replacing A by $N\tau\text{-}cl_\theta(A)$.

Remark 3.8 $N\tau\text{-}cl_\theta(A) \neq N\tau\text{-}cl_\theta(N\tau\text{-}cl_\theta(A))$. That is, $N\tau\text{-}cl_\theta(A)$ is not $N\tau\theta$-closed set. For example let $N = 4$, $X = \{a,b,c,d,e\}$, consider $\tau_1O(X) = \{\varnothing, X, \{a,b\}\}$, $\tau_2O(X) = \{\varnothing, X, \{a,b,c,d\}\}$, $\tau_3O(X) = \{\varnothing, X, \{c,d\}\}$ and $\tau_4O(X) = \{\varnothing, X, \{c,d\}, \{a,b,c,d\}\}$. Then $4\tau O(X) = \{\varnothing, X, \{a,b\}, \{c,d\}, \{a,b,c,d\}\}$ is a 4-topology. If $A = \{a\} \subset X$, then $4\tau\text{-}cl_\theta(A) = \{a,b,c\}$ and $4\tau\text{-}cl_\theta(4\tau\text{-}cl_\theta(A)) = X$. Thus, $4\tau\text{-}cl_\theta(A) \neq 4\tau\text{-}cl_\theta(4\tau\text{-}cl_\theta(A))$.

Theorem 3.9 $N\tau\text{-}cl_\theta(A)$ is $N\tau$-closed, for any subset A of X.

Proof: We know that $N\tau\text{-}cl_\theta(A) \subseteq N\tau\text{-}cl(N\tau\text{-}cl_\theta(A))$ and it is enough to prove $N\tau\text{-}cl(N\tau\text{-}cl_\theta(A)) \subseteq N\tau\text{-}cl_\theta(A)$. Let $x \in N\tau\text{-}cl(N\tau\text{-}cl_\theta(A))$, then $N\tau\text{-}cl_\theta(A) \cap G \neq \varnothing$ for every $N\tau$-open set G containing x. Let $y \in N\tau\text{-}cl_\theta(A) \cap G$, then $y \in G$ and $y \in N\tau\text{-}cl_\theta(A)$. Since G is $N\tau$-open set containing y and $y \in N\tau\text{-}cl_\theta(A)$, $A \cap N\tau\text{-}cl(G) \neq \varnothing$. Therefore, $x \in N\tau\text{-}cl_\theta(A)$ and so $N\tau\text{-}cl_\theta(A)$ is $N\tau$-closed.

Theorem 3.10 Every $N\tau\theta$-closed set is $N\tau$-closed.

Proof: Since $N\tau\text{-}cl_\theta(A) = A$ and $N\tau\text{-}cl_\theta(A)$ is $N\tau$-closed, then A is $N\tau$-closed.

Theorem 3.11 For subsets A and B of a topological space $(X, N\tau)$.

1. $A \subseteq B \Rightarrow N\tau\text{-}cl_\theta(A) \subseteq N\tau\text{-}cl_\theta(B)$.
2. $N\tau\text{-}cl_\theta(A \cup B) = N\tau\text{-}cl_\theta(A) \cup N\tau\text{-}cl_\theta(B)$.
3. $N\tau\text{-}cl_\theta(A \cap B) \subseteq N\tau\text{-}cl_\theta(A) \cap N\tau\text{-}cl_\theta(B)$.

Proof: 1. If $A \subseteq B$ and $x \in N\tau\text{-}cl_\theta(A)$, $A \cap N\tau\text{-}cl(G) \neq \emptyset$ for every $N\tau$-open set G containing x. Since $A \cap N\tau\text{-}cl(G) \subseteq B \cap N\tau\text{-}cl(G)$, $B \cap N\tau\text{-}cl(G) \neq \emptyset$ for every $N\tau$-open set G containing x and hence $x \in N\tau\text{-}cl_\theta(B)$. Thus, $N\tau\text{-}cl_\theta(A) \subseteq N\tau\text{-}cl_\theta(B)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, using (1), $N\tau\text{-}cl_\theta(A) \subseteq N\tau\text{-}cl_\theta(A \cup B)$ and $N\tau\text{-}cl_\theta(B) \subseteq N\tau\text{-}cl_\theta(A \cup B)$. Therefore, $N\tau\text{-}cl_\theta(A) \cup N\tau\text{-}cl_\theta(B) \subseteq N\tau\text{-}cl_\theta(A \cup B)$. On the other hand, if $x \in N\tau\text{-}cl_\theta(A \cup B)$, then $(A \cup B) \cap (N\tau\text{-}cl(G)) \neq \emptyset$ for every $N\tau$-open set G containing x. That is, $[A \cap N\tau\text{-}cl(G)] \cup [B \cap N\tau\text{-}cl(G)] \neq \emptyset$ and therefore, $A \cap N\tau\text{-}cl(G) \neq \emptyset$ or $A \cap N\tau\text{-}cl(G) \neq \emptyset$. That is, $x \in N\tau\text{-}cl_\theta(A)$ or $x \in N\tau\text{-}cl_\theta(B)$ and hence, $x \in N\tau\text{-}cl_\theta(A) \cup N\tau\text{-}cl_\theta(B)$. Thus, $N\tau\text{-}cl_\theta(A \cup B) \subseteq N\tau\text{-}cl_\theta(A) \cup N\tau\text{-}cl_\theta(B)$. Hence $N\tau\text{-}cl_\theta(A \cup B) = N\tau\text{-}cl_\theta(A) \cup N\tau\text{-}cl_\theta(B)$.

3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (1), $N\tau\text{-}cl_\theta(A \cap B) \subseteq N\tau\text{-}cl_\theta(A)$ and $N\tau\text{-}cl_\theta(A \cap B) \subseteq N\tau\text{-}cl_\theta(B)$. Therefore, $N\tau\text{-}cl_\theta(A \cap B) \subseteq N\tau\text{-}cl_\theta(A) \cap N\tau\text{-}cl_\theta(B)$

Example 3.12 Let $X = \{a,b,c,d\}$, then $\tau_1O(X) = \{\emptyset, X, \{a\}\}$, $\tau_2O(X) = \{\emptyset, X, \{b,d\}\}$, $\tau_3O(X) = \{\emptyset, X, \{a,b,d\}\}$. Then $3\tau O(X) = \{\emptyset, X, \{a\}, \{a,b,d\}, \{b,d\}\}$ is a tritopology. Let $A = \{a,b\}$ and $B = \{a,c\}$ Then $3\tau\text{-}cl_\theta(A) = X$ and $3\tau\text{-}cl_\theta(B) = X$. Therefore, $3\tau\text{-}cl_\theta(A \cap 3\tau\text{-}cl_\theta(B) = X$ But $3\tau\text{-}cl_\theta(A \cap B) = 3\tau\text{-}cl_\theta(\{a\}) = \{a,c\}$. Therefore $3\tau\text{-}cl_\theta(A \cap B) \neq 3\tau\text{-}cl_\theta(A) \cap 3\tau\text{-}cl_\theta(B)$. That is equality does not hold in (3) of previous theorem.

Corollary 3.13 If A and B are $N\tau\theta$-closed, then $A \cup B$ is $N\tau\theta$-closed.

Proof: $N\tau\text{-}cl_\theta(A) = A$ and $N\tau\text{-}cl_\theta(B) = B$, since A and B are $N\tau\theta$-closed. Therefore $N\tau\text{-}cl_\theta(A \cup B)$ = $N\tau\text{-}cl_\theta(A) \cup N\tau\text{-}cl_\theta(B)$ = $A \cup B$ and hence, $A \cup B$ is $N\tau\theta$-closed.

Corollary 3.14 If A and B are $N\tau\theta$-closed, then $A \cap B$ is $N\tau\theta$-closed.

Proof: $N\tau\text{-}cl_\theta(A) = A$ and $N\tau\text{-}cl_\theta(B) = B$, since A and B are $N\tau\theta$-closed. Therefore $N\tau\text{-}cl_\theta(A \cap B) \subseteq N\tau\text{-}cl_\theta(A) \cap N\tau\text{-}cl_\theta(B) = A \cap B$ and hence, $A \cap B$ is $N\tau\theta$-closed.

Remark 3.15 From the above two corollaries it is very clear that $N\tau\theta$-closed sets form a topology.

Theorem 3.16 Every $N\tau\theta$-open set is $N\tau$-open in a N-topological space.

Proof: If A is a $N\tau\theta$-open in $(X, N\tau)$, then $X - A$ is $N\tau\theta$-closed. If $x \in N\tau\text{-}cl(X - A)$, $x \in N\tau\text{-}cl_\theta(X - A)$, since $N\tau\text{-}cl(X - A) \subseteq N\tau\text{-}cl_\theta(X - A)$. But $N\tau\text{-}cl_\theta(X - A) = X - A$. Therefore, $x \in X - A$. Thus, $x \in N\tau\text{-}cl(X - A) \Rightarrow x \in X - A$. Therefore, $N\tau\text{-}cl(X - A) \subseteq X - A$. But $X - A \subseteq N\tau\text{-}cl(X - A)$. Hence, $N\tau\text{-}cl(X - A) = X - A$. That is, $X - A$ is $N\tau$-closed and A is $N\tau$-open in X. Thus any $N\tau\theta$-open set is $N\tau$-open. That is the $N\tau\theta$-open sets is stronger than the $N\tau$-open sets.

Remark 3.17 The converse of the above theorem need not be true. For example, N=4, $X=\{a,b,c,d,\}$, then $\tau_1 O(X)=\{\varnothing, X,\{a\}\}$, $\tau_2 O(X)=\{\varnothing, X,\{b\}\}$, $\tau_3 O(X)=\{\varnothing, X,\{a,b\}\}$ and $\tau_4 O(X)=\{\varnothing, X,\{c\}\}$. Then $4\tau O(X)=\{\varnothing, X,\{b,d\}\}$ is a 4-topology. Let $A=\{b,d\}$ is 4τ-open in X. But $X-A=\{a,c\}$; $4\tau\text{-}cl_\theta(X-A)=X$ and hence $X-A$ is not $4\tau\theta$-closed. Therefore, A is not $4\tau\theta$-open in X.

θ-interior in N-Topological Spaces: *In this section we derive θ-interior operator in N-topology and also establish its properties.*

Definition 4.1 Let $A\subseteq(X,N\tau)$ then an element $x\in A$ is said to be a $N\tau\theta$-interior point of A if $N\tau\text{-}cl(G)\subseteq A$ for some $N\tau$-open set G containing x.The set of all $N\tau\theta$-interior points of A is called the $N\tau\theta$-interior of A and is denoted by $N\tau\text{-}int_\theta(A)$.

Example 4.2 Let $X=\{a,b,c,d,e\}$ and $N=3$, then $\tau_1 O(X)=\{\varnothing, X,\{c\}\}$, $\tau_2 O(X)=\{\varnothing, X,\{a,b\}\}$ and $\tau_3 O(X)=\{\varnothing, X,\{a,b,c\}\}$. Then $3\tau O(X)=\{\varnothing, X,\{c\},\{a,b\},\{a,b,c\}\}$ is a tritopology. Let $A=\{c,d,e\}$. The element c is $3\tau\theta$-interior point of A, since $\{c\}$ is a 3τ-open set containing c and $3\tau\text{-}cl(\{c\})=\{c,d,e\}\subseteq A$. The element d is not $3\tau\theta$-interior point of A, since X is the only 3τ-open set containing d and 3τ-$cl(X)=X$ŮA. Similarly the element e is not $3\tau\theta$-interior point of A. Thus, $3\tau\text{-}int_\theta(A)=\{c\}$.

Theorem 4.3 $N\tau\text{-}int_\theta(A)\subseteq N\tau\text{-}int(A)$ for any $A\subseteq(X,N\tau)$.

Proof: If $x\in N\tau\text{-}int_\theta(A)$ then $N\tau\text{-}cl(G)\subseteq A$ for some $N\tau$-open set G containing x. Also $G\subseteq N\tau\text{-}cl(G)\subseteq A$. Thus, G is a $N\tau$-open subset A containing x such that $G\subseteq A$. Therefore, $x\in N\tau\text{-}int(A)$, since $N\tau\text{-}int(A)$ is the largest $N\tau$-open subset A. Thus, $N\tau\text{-}int_\theta(A)\subseteq$ $N\tau$-$int(A)$.

Theorem 4.4 In a $N\tau$ topological space $(X,N\tau)$,

1. $X-N\tau\text{-}int_\theta(A)=N\tau\text{-}cl_\theta(X-A)$.
2. $X-N\tau\text{-}cl_\theta(A)=N\tau\text{-}int_\theta(X-A)$. **Proof:** 1. $x\in X-N\tau\text{-}int_\theta(A)$ if and only if $x\notin N\tau$-$int_\theta(A)$ if and only if $N\tau\text{-}cl(G)$ŮA for every $N\tau$-open set G containing x iff $N\tau$-$cl(G)\cap(X-A)\neq\varnothing$ for every $N\tau$-open set G containing x if and only if $x\in N\tau\text{-}cl_\theta(X-A)$. Thus, $X-N\tau\text{-}int_\theta(A)=N\tau\text{-}cl_\theta(X-A)$.
2. Proof is similar to that of (1).

Theorem 4.5 If $A,B\subseteq(X,N\tau)$, then

1. $N\tau\text{-}int_\theta(A)\subseteq A$.
2. $N\tau\text{-}int_\theta(A)=\cup\{G:G$ is $N\tau$-open and $N\tau\text{-}cl(G)\subseteq A\}$.
3. A is $N\tau\theta$-open if and only if $A=N\tau$ $\text{-}int_\theta(A)$.
4. $A\subseteq B\Rightarrow N\tau\text{-}int_\theta(A)\subseteq N\tau\text{-}int_\theta(B)$.

5. $N\tau\text{-}int_\theta(N\tau\text{-}int_\theta(A)) \subseteq N\tau\text{-}int_\theta(A)$.
6. $N\tau\text{-}int_\theta(A) \cup N\tau\text{-}int_\theta(B) \subseteq N\tau\text{-}int_\theta(A \cup B)$.
7. $N\tau\text{-}int_\theta(A \cap B) = N\tau\text{-}int_\theta(A) \cap N\tau\text{-}int_\theta(B)$.

Proof: 1. $N\tau\text{-}int_\theta(A) \subseteq N\tau\text{-}int(A) \subseteq A$ and hence $N\tau\text{-}int_\theta(A) \subseteq A$.

2. $x \in N\tau\text{-}int_\theta(A)$ if and only if $N\tau\text{-}cl(G) \subseteq A$ for some $N\tau$-open set G containing x if and only if $x \in \cup\{G : G$ is $N\tau$-open containing x and $N\tau\text{-}cl(G) \subseteq A\}$. Thus $N\tau\text{-}int_\theta(A) = \cup\{G : G$ is $N\tau$-open and $N\tau\text{-}cl(G) \subseteq A\}$.

3. A is $N\tau\theta$-open if and only if $X - A$ is $N\tau\theta$-closed if and only if $N\tau\text{-}cl_\theta(X - A) = X - A$ if and only if $X - N\tau\text{-}int_\theta(A) = X - A$ if and only if $N\tau\text{-}int_\theta(A) = A$.

4. if $A \subseteq B$ and $x \in N\tau\text{-}int_\theta(A)$, then $N\tau\text{-}cl(G) \subseteq A \subseteq B$ for some $N\tau$-open set G containing x and hence $x \in N\tau\text{-}int_\theta(B)$. Therefore, $N\tau\text{-}int_\theta(A) \subseteq N\tau\text{-}int_\theta(B)$.

5. Since $N\tau\text{-}int_\theta(A) \subseteq A$, by (4), $N\tau\text{-}int_\theta(N\tau\text{-}int_\theta(A) \subseteq N\tau\text{-}int_\theta(A)$

6. Since, $A, B \subseteq A \cup B$, by (4), $N\tau\text{-}int_\theta(A) \cup N\tau\text{-}int_\theta(B) \subseteq N\tau\text{-}int_\theta(A \cup B)$.

7. Since $X - N\tau\text{-}int_\theta(A \cap B) = N\tau\text{-}cl_\theta(X - A \cap B)$, $N\tau\text{-}int_\theta(A \cap B) = N\tau\text{-}int_\theta(A) \cap N\tau\text{-}int_\theta(B)$.

Remark 4.6 Equality does not hold in (1) and (6). Let $X = \{a,b,c,d\}$ and $N = 2$, then $\tau_1 O(X) = \{\varnothing, X, \{a\}, \{b,d\}\}$ and $\tau_2 O(X) = \{\varnothing, X, \{a,b,d\}\}$. Then $2\tau O(X) = \{\varnothing, X, \{a\}, \{b,d\}, \{a,b,d\}\}$ is a bitopology. Let $A = \{a,b,c\}$ and $B = \{b,c,d\}$. Then $2\tau\text{-}int_\theta(A) = \{a\}$ and $2\tau\text{-}int_\theta(B) = \{b,d\}$. Therefore, $2\tau\text{-}int_\theta(A) \cup 2\tau\text{-}int_\theta(B) = \{a,b,d\}$, but $2\tau\text{-}int_\theta(A \cup B) = X$. That is, $2\tau\text{-}int_\theta(A \cup B) \neq 2\tau\text{-}int_\theta(A) \cup 2\tau\text{-}int_\theta(B)$. Also $A \neq 2\tau\text{-}int_\theta(A)$.

Theorem 4.7 If $N\tau\theta O(X)$ denotes the set of all $N\tau\theta$-open sets in $(X, N\tau)$, then $N\tau\theta O(X)$ is a topology on X.

Proof:

1. Since $\varnothing$ and X are $N\tau\theta$-closed, they are $N\tau\theta$-open and hence $\varnothing$ and $X \in N\tau\theta O(X)$.

2. If $A_i \in N\tau\theta O(X)$, then each A_i is $N\tau\theta$-open in X and hence $N\tau\text{-}int_\theta(A_i) = A_i$ for each i. Let $A = \cup_i A_i$. Consider $N\tau\text{-}int_\theta(A) = N\tau\text{-}int_\theta(\cup_i A_i) \supseteq \cup_i \ N\tau\text{-}int_\theta(A_i \text{-} \cup_i A_i = A$. That is $A \subseteq N\tau\text{-}int_\theta(A)$. But $N\tau\text{-}int_\theta(A) \subseteq A$. Therefore $N\tau\text{-}int_\theta(A) = A$. Thus A is $N\tau\theta$-open. Thus arbitrary union of members of $N\tau\theta O(X)$ belong to $N\tau\theta O(X)$.

3. Let A and $B \in N\tau\theta O(X)$. Then A, B are $N\tau\theta$-open and hence $N\tau\text{-}int_\theta(A) = A$ and $N\tau\text{-}int_\theta(B) = B$. Consider $N\tau\text{-}int_\theta(A \cap B) = N\tau\text{-}int_\theta(A) \cap N\tau\text{-}int_\theta(B) = A \cap B$ and therefore, $A \cap B$ is $N\tau\theta$-open. That is, $A \cap B \in N\tau\theta O(X)$ whenever $A, B \in N\tau\theta O(X)$. Thus $N\tau\theta O(X)$ is a topology on X

Remark 4.8 Since any $N\tau\theta$-open set is $N\tau\theta$-open, $N\tau\theta O(X) \subseteq N\tau O(X)$ and hence $N\tau\theta O(X)$ is stronger than $N\tau O(X)$.

Theorem 4.9 If $N\tau O(X)=\{X,\varnothing\}$ then $N\tau\text{-}int_{\theta}(A)=\varnothing$ for every proper subset A of X. $Proof: Let$ $A\subset X$ and $x\in A$. Since any $N\tau$-open set containing x is X and $N\tau\text{-}cl(X)=$ XÚA, x is not a $N\tau\theta$-interior point of A. That is no element of A is a $N\tau\theta$-interior point of itself. Therefore, $N\tau\text{-}int_{\theta}(A)=\varnothing$ for every $A\subset X$.

Conclusion: Having been defined the strong form of generalized closed sets in N-topology in terms of $N\tau\theta$-closure we see that this can be extended to other areas like neutrosophic, digital, fuzzy sets etc. To add strength to this theory we also have illustrated a few examples here. Further we hope that this concept can pave way to many other research fields.

References:

1. S.Jafari, Some properties of quasi-continuous functions• , Far East J. Math. Sci, vol. 6 , 1998, pp. 689-696.
2. J.C.Kelly,Bitopologicalspaces, Proceedings London Mathematical Society, vol.3, 1963, pp. 71-89.
3. M.Lellis Thivagar, V.Ramesh and M.Arockia Dasan, On new structure of N-topology, Cogent Mathematics (Taylor and Francis), vol. 3, 2016. UGC J.No: 48730.
4. T.Noiri, Properties of continuous functions• , Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur, vol. 58, 1975, pp. 887-891.
5. N.V.Velicko, H-closed topological spaces, Trans. Amer. Math. Soc., vol. 78, 1968, pp. 103-118.

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