STRONGLY MULTIPLICATIVE LABELING OF CYCLE C_n WITH PARALLEL P_3 CHORDS

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Abstract: Labeled graphs serve as useful models for a broad range of applications such as coding theory, missile guidance codes, convolution codes and communication network. A graph G with p vertices is said to be strongly multiplicative if the vertices of G are labeled with distinct integers 1,2,3,....p such that the labels induced on the edges by the product of the end vertices are distinct. The main aim of this paper is to show that cycles C_n with parallel P_3 chords admit strongly multiplicative labeling.

Keywords: Chord, Cycles with Parallel P_3 Chords, Labeling, Strongly Multiplicative Labeling.

Introduction: The graph considered here is a finite simple undirected graph $G = (V, E)$ where $|V(G)|=p, |E(G)|=q$. A graph labeling is an assignment of integers to the vertices (or) edges (or) both subject to certain conditions. An extensive survey on graph labeling can be found in Gallian [2]. Beineke and Hegde [1] call a graph G with p vertices strongly multiplicative if the vertices of G are labelled with distinct integers 1,2,.......p such that the labels induced on the edges by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees, cycles, wheels, K_n if and only if ≤ 5 , $K_{r,r}$ if and only if $r \leq 4$,P_m X P_n. Seoud and Zid [6] prove the following graphs are strongly multiplicative rK_n for all r and n atmost 5; $K_{4,r}$ for all r; corona of P_n and K_m for all n and $2 \leq m \leq 8$. Germina and Ajitha [3] proved that Peterson graphs, Quadrilateral snakes and ladders are strongly multiplicative. Muthusamy, Rajasekar and BaskarBabujee [5] proved strongly multiplicative labeling of cycle related graphs. Kanani and Chhaya [4] have proved strongly multiplicative labeling of some path related graphs. Vaidya and Kanani [8] have shown that cycle with one chord, cycle with twin chords, cycle with triangle, Duplication of an arbitrary vertex of cycle C_n are strongly multiplicative.

A cycle on n vertices is denoted by C_n and a path of k vertices is denoted by P_k. A chord (path) of a cycle is an edge connecting two otherwise non-adjacent vertices of a cycle. In this paper we prove strongly multiplicative labeling of cycle C_n with parallel P_3 chords.

Definition 1.1: A graph G is called a cycle with parallel P_3 chords [7] if G is obtained from the cycle C_n: $u_0u_1u_2......u_{n-1}u_0$ by adding disjoint paths P_3's between the pair of vertices u_1u_{n-1}, u_2u_{n-2},... $u_\propto u_\beta$ of C_n where $\propto = \left\lfloor \frac{n}{2} \right\rfloor - 1$, $\beta = \left\lfloor \frac{n}{2} \right\rfloor + 2$ if n is odd (or) $\beta = \left\lfloor \frac{n}{2} \right\rfloor + 1$ if n is even as shown in Fig. 1a and 1b. Then G has $\frac{3n-3}{2}$ vertices and $2n-3$ edges if n is odd and $\frac{3n-2}{2}$ vertices and $2n-2$ edges if n is even.

Definition 1.2: A graph G with p vertices is said to be strongly multiplicative if the vertices are assigned distinct numbers 1,2,3,.....p such that the labels induced on the edges by the product of the end vertices are distinct.

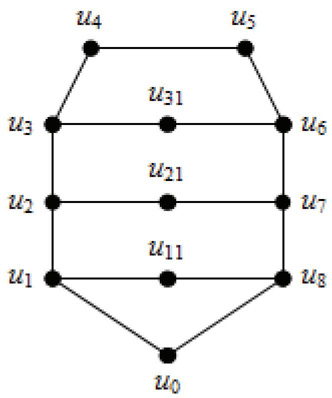

Fig. 1a Cycle C_9 with Parallel P_3 chords

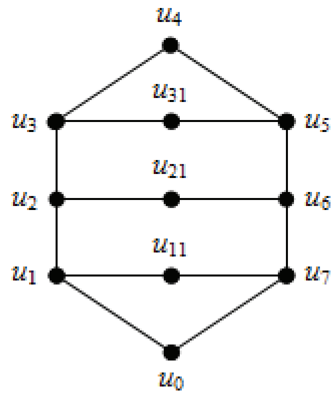

Fig. 1b Cycle C_8 with Parallel P_3 chords

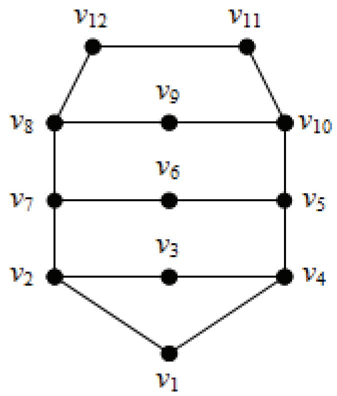

Fig. 2a Cycle C_9 with Parallel P_3 chords having Hamiltonian path starting at v_1 and ending at v_{12}

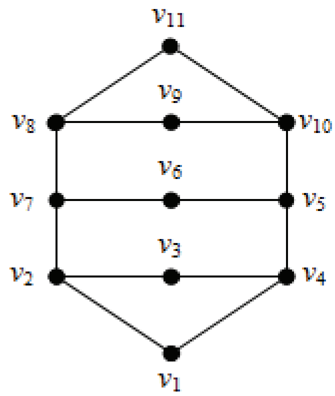

Fig. 2b Cycle C_8 with Parallel P_3 chords having Hamiltonian path starting at v_1 and ending at v_{11}

Main Result:

Theorem 2.1 Every cycle C_n ($n \geq 8$) with parallel P_3 chords admit strongly multiplicative labeling.

Proof: Consider the graph G as a cycle C_n with parallel P_3 chords. Using definition 1.1 G is obtained from the cycle C_n: $u_0u_1u_2......u_{n-1}u_0$ by adding disjoint paths P_3's between each pair of vertices u_1u_{n-1}, u_2u_{n-2},........ $u_\propto u_\beta$ of C_n where $\propto = \left\lfloor\frac{n}{2}\right\rfloor - 1$, $\beta = \left\lfloor\frac{n}{2}\right\rfloor + 2$ if n is odd (or) $\beta = \left\lfloor\frac{n}{2}\right\rfloor + 1$ if n is even. Let v_1, v_2, v_3, v_N be the vertices of G. Here G has a Hamiltonian path starting at $v_1 = u_0$ and ending at $v_N = u_{\left\lfloor\frac{n}{2}\right\rfloor}$ of cycle C_n of G as shown in Fig 2a and 2b. The Hamiltonian path of G is $v_1 v_2 v_3 ... v_N$ where $N = |V(G)|$. Two cases are considered here depending on the parameter n.

Case 1: If n is even

$|V(G)| = N = \frac{3n-2}{2}$ and $|E(G)| = 2n - 2$. $v_1, v_2, ...v_N$ are the N vertices of G. Define a bijection f: $V(G) \rightarrow \{1,2,3,.....N\}$ as follows:

Subcase 1.1: If $n \equiv 0 \pmod 4$

$$f(v_i) = \begin{cases} i, & i = 1,5,9,\dots,N-2 \quad \text{if } n = \text{multiple of } 8 \\ i, & i = 1,5,9,\dots,N \quad\quad \text{if } n \neq \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i+1, & i = 2,6,10,\dots,N-1 \text{ if } n = \text{multiple of } 8 \\ i+1, & i = 2,6,10,\dots,N-2 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$f(v_i) = \begin{cases} i-1, & i = 3,7,11,\dots,..N \text{ if } n = \text{multiple of } 8 \\ i-1, & i = 3,7,11,\dots,N-3 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$f(v_i) = \begin{cases} i, & i = 4,8,12,\dots,N-3 \text{ if } n = \text{multiple of } 8 \\ i, & i = 4,8,12,\dots,N-1 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
From the above labeling it is clear that f is a bijection from $V(G) \rightarrow \{1,2,3,\dots..N\}$.

The edge set E of G is given by $E = E_1 \cup E_2 \cup E_3 \cup \dots\dots \cup E_9$ where
$$E_1 = \begin{cases} v_i v_{i+1}, & i = 1,5,9,\dots,N-2 \text{ if } n = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 1,5,9,\dots,N-4 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$E_2 = \begin{cases} v_i v_{i+1}, & i = 2,6,10,\dots,N-1 \text{ if } n = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 2,6,10,\dots,N-3 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$E_3 = \begin{cases} v_i v_{i+1}, & i = 3,7,11,\dots,N-4 \text{ if } n = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 3,7,11,\dots,N-2 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$E_4 = \begin{cases} v_i v_{i+1}, & i = 4,8,12,\dots,N-3 \text{ if } n = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 4,8,12,\dots,N-1 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
are the edges along the Hamiltonian path and
$$E_5 = \left\{ v_{12i-10} v_{12i-5}, 1 \le i \le \left\lfloor \frac{n}{8} \right\rfloor \right\}$$
$$E_6 = \left\{ v_{12i-4} v_{12i+1}, 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor \right\}$$
$$E_7 = \left\{ v_{12i-7} v_{12i-2}, 1 \le i \le \left\lfloor \frac{n}{8} \right\rfloor \right\}$$
$$E_8 = \left\{ v_{12i-1} v_{12i+4}, 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor \right\}$$
$$E_9 = \{ v_1 v_4, v_{N-3} v_N \}$$
are the edges that are not in the Hamiltonian path.

Define the induced function $f^*: E(G) \rightarrow I$ (set of positive integers) as $f^*(v_i v_j) = f(v_i) f(v_j)$ for all $v_i v_j \in E$ and $v_i, v_j \in V$. The edge labeling for the edges along the Hamiltonian path are given by
$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+2), & i = 1,5,9,\dots\dots N-2 \text{ if } n = \text{multiple of } 8 \\ i\,(i+2), & i = 1,5,9,\dots\dots N-4 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+1), & i = 2,6,10,\dots\dots N-1 \text{ if } n = \text{multiple of } 8 \\ i\,(i+1), & i = 2,6,10,\dots\dots N-3 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$f^*(v_i v_{i+1}) = \begin{cases} (i-1)(i+1), & i = 3,7,11,\dots\dots N-4 \text{ if } n = \text{multiple of } 8 \\ (i-1)\,(i+1), & i = 3,7,11,\dots\dots N-2 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+1), & i = 4,8,12,\dots\dots N-3 \text{ if } n = \text{multiple of } 8 \\ i\,(i+1), & i = 4,8,12,\dots\dots N-1 \text{ if } n \neq \text{multiple of } 8 \end{cases}$$
The edge labeling for the edges that are not in the Hamiltonian path are given by $f^*(v_{12i-10} v_{12i-5}) = (12i-9)(12i-6),\ 1 \le i \le \left\lfloor \frac{n}{8} \right\rfloor$
$$f^*(v_{12i-4} v_{12i+1}) = (12i-4)(12i+1),\ 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor$$
$$f^*(v_{12i-7} v_{12i-2}) = (12i-7)(12i-1),\ 1 \le i \le \left\lfloor \frac{n}{8} \right\rfloor$$
$$f^*(v_{12i-1} v_{12i+4}) = (12i-2)(12i+4),\ 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor$$
$$f^*(v_1\ v_4) = 4$$
$$f^*(v_{N-3} v_N) = \begin{cases} (N-1)(N-3) \text{ if } n = \text{multiple of } 8 \\ (N-2)N \quad \text{ if } n \neq \text{multiple of } 8 \end{cases}$$

To show that the edge labels within the edge sets and among the edge sets are distinct we will prove them by assuming that they are same and arrive at a contradiction. We have proved for the case $n =$ multiple of 8 and in a similar way it can be proved for $n \neq$ multiple of 8.
Consider the edge sets E_1, E_2, E_3, E_4 along the Hamiltonian path:
To show that the edge labels within the edge sets are distinct
For the edges in E_1: If $i \neq$ j,

$i = 1,5,9, \dots, N-2$, $j = 1,5,9, \dots, N-2$
Let $f^*(v_i v_{i+1}) = f^*(v_j v_{j+1})$
⇨ $i(i+2) = j(j+2)$
gives $i+j = -2$, a contradiction.

Hence the edge labels of E_1 are distinct. In a similar way it can be shown that the edge labels of E_2, E_3, E_4 are distinct.
To show that the edge labels of different edge sets are distinct, it is observed that the edge labels of E_1 are odd and the edge labels of E_2, E_3, E_4 are even.
For the edges of E_2 and E_4 :
If $i \neq$ j, $i = 2,6,10, \dots, N-1$,
$j = 4,8,12, \dots, N-3$
Let $f^*(v_i v_{i+1}) = f^*(v_j v_{j+1})$
⇨ $(i+1)\, i = (j+1)\, j$
gives $i+j = -1$, a contradiction.

Hence the edge labels of E_2 and E_4 are distinct. In a similar way it can be shown that the edge labels of E_2, E_3 and E_3, E_4 are distinct.
Consider the edge sets E_5, E_6, E_7, E_8 that are not in the Hamiltonian path
To show that the edge labels within the edge sets are distinct
For the edges in E_5:
If $i \neq$ j,$1 \leq i \leq \left\lfloor \frac{n}{8} \right\rfloor$,$1 \leq j \leq \left\lfloor \frac{n}{8} \right\rfloor$
Let $f^*(v_i v_{i+1}) = f^*(v_j v_{j+1})$
⇨ $(12i-9)(12i-6) = (12j-9)(12j-6)$
Gives $i=j$, a contradiction.

Hence the edge labels of E_5 are distinct. In a similar way it can be shown that the edge labels of E_6, E_7, E_8 are distinct.
To show that the edge labels of different edge sets are distinct, it is observed that the edge labels of E_7 are odd and the edge labels of E_5, E_6, E_8 are even.
For the edges of E_5 and E_6:
If $i \neq$ j,$1 \leq i \leq \left\lfloor \frac{n}{8} \right\rfloor$, $1 \leq j \leq \left\lfloor \frac{n-4}{8} \right\rfloor$
Let $f^*(v_{12i-10} v_{12i-5}) = f^*(v_{12j-4} v_{12j+1})$
⇨ $(12i-9)(12i-6) = (12j-4)(12j+1)$
⇨ $i^2 - j^2 = \frac{18(5i-j)-29}{72}$
Gives left hand side as an integer and right hand side not as an integer, a contradiction. Hence the edge labels of E_5 and E_6 are distinct. In a similar way it can be shown that the edge labels of E_5, E_8 and E_6, E_8 are distinct.
For the edges of E_9:
$(N-1)(N-3) \neq 4$ as $N \geq 11$.

Hence the edge labels of E_9 are distinct.
Among the edge sets that are in the Hamiltonian path and not in the Hamiltonian path, it is seen that edge labels of E_1 and E_7 are odd and the edge labels of E_2,E_3, E_4, E_5, E_6, E_8 are even.
For the edges of E_1 and E_7:
If $i \neq$ j , $i = 1,5,9, \dots, N-2$, $1 \leq j \leq \left\lfloor \frac{n}{8} \right\rfloor$
Let $f^*(v_i v_{i+1}) = f^*(v_{12j-7} v_{12j-2})$
⇨ $(i+2)\, i = (12j-7)(12j-1)$
⇨ $(i+1)^2 = 16(3j-1)^2 - 8$
⇨ $\frac{(i+1)^2}{8} = 2(3j-1)^2 - 1$
Shows that left hand side is not an integer but right hand side is an integer, a contradiction. Hence the edge labels of E_1 and E_7 are distinct.

For the edges of E_2 and E_6:
If $i \neq j$, $i = 2,6,10,\ldots,N-1$,
$1 \leq j \leq \left\lfloor \frac{n-4}{8} \right\rfloor$
Let $f^*(v_i v_{i+1}) = f^*(v_{12j-4} v_{12j+1})$
⇨ $(i+1)\, i = (12j-4)(12j+1)$
⇨ $\frac{(i+1)i}{4} = 36\, j^2 - 9j - 1$
Shows that left hand side is not an integer but right hand side is an integer, a contradiction. Hence the edge labels of E_2 and E_6 are distinct. The proof follows as above for the edge sets E_2,E_5 and E_2 ,E_8.

For the edges of E_3 and E_5:
If $i \neq j$, $i = 3,7,11,\ldots,N-4$,
$1 \leq j \leq \left\lfloor \frac{n}{8} \right\rfloor$
Let $f^*(v_i v_{i+1}) = f^*(v_{12j-10} v_{12j-5})$
⇨ $i^2 - 1 = 18(8j^2 - 10j + 3)$
⇨ $\frac{i^2-1}{18} = 8j^2 - 10j + 3$

Shows that left hand side is not an integer, but right hand side is an integer, a contradiction. Hence the edge labels of E_3 and E_5 are distinct. The proof follows as above for the edge sets E_3 , E_6 and E_3 , E_8.

For the edges of E_4 and E_6**:**
If $i \neq j$, $i = 4,8,12,\ldots,N-3$,$1 \leq j \leq \left\lfloor \frac{n-4}{8} \right\rfloor$
Let $f^*(v_i v_{i+1}) = f^*(v_{12j-4} v_{12j+1})$
⇨ $(i+1)\, i = (12j-4)(12j+1)$
⇨ $(i+\frac{1}{2})^2 - \frac{1}{4} = \frac{9}{4}(8j-1)^2 - \frac{25}{4}$
⇨ $\frac{(2i+1)^2+24}{9} = (8j-1)^2$
Shows that left hand side is not an integer, but right hand side is an integer, a contradiction. Hence the edge labels of E_4 and E_6 are distinct. The proof follows as above for the edge sets E_4 ,E_5 and E_4 , E_8. Thus all the edges of G have distinct labels. Hence the graph under consideration admits strongly multiplicative labeling. An illustration is given in Fig 3a.

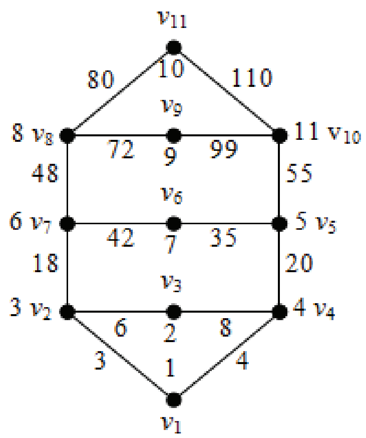

Fig. 3a Strongly multiplicative labeling of Cycle C_8 with Parallel P_3 chords

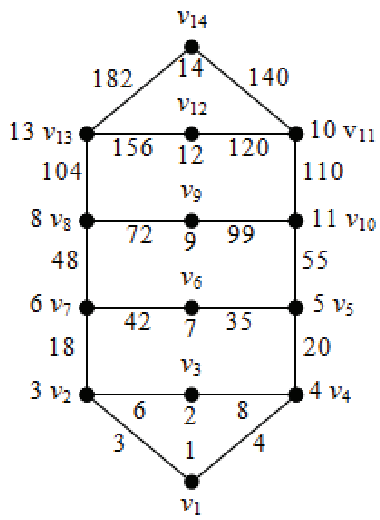

Fig. 3b Strongly multiplicative labeling of Cycle C_{10} with Parallel P_3 chords

Subcase 1.2: If $n \equiv 2 \pmod 4$

$$f(v_i) = \begin{cases} i, & i = 1,5,9,\dots,N-1 \quad \text{if } n-2 = \text{multiple of } 8 \\ i, & i = 1,5,9,\dots,N-3 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i+1, & i = 2,6,10,\dots,N-4 \quad \text{if } n-2 = \text{multiple of } 8 \\ i+1, & i = 2,6,10,\dots,N-2 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i-1, & i = 3,7,11,\dots,N-3 \quad \text{if } n-2 = \text{multiple of } 8 \\ i-1, & i = 3,7,11,\dots,N-1 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i, & i = 4,8,12,\dots,N-2 \quad \text{if } n-2 = \text{multiple of } 8 \\ i, & i = 4,8,12,\dots,N \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$f(v_N) = N$ if $n-2 = $ multiple of 8

From the above labeling it is clear that f is a bijection from $V(G) \rightarrow \{1,2,3,\dots,N\}$.
The edge set E of G is given by $E = E_1 \cup E_2 \cup E_3 \cup \dots \cup E_9$ where

$$E_1 = \begin{cases} v_i v_{i+1}, & i = 1,5,9,\dots,N-1 \quad \text{if } n-2 = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 1,5,9,\dots,N-3 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$E_2 = \begin{cases} v_i v_{i+1}, & i = 2,6,10,\dots,N-4 \quad \text{if } n-2 = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 2,6,10,\dots,N-2 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$E_3 = \begin{cases} v_i v_{i+1}, & i = 3,7,11,\dots,N-3 \quad \text{if } n-2 = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 3,7,11,\dots,N-1 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$E_4 = \begin{cases} v_i v_{i+1}, & i = 4,8,12,\dots,N-2 \quad \text{if } n-2 = \text{multiple of } 8 \\ v_i v_{i+1}, & i = 4,8,12,\dots,N-4 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

are the edges along the Hamiltonian path and

$$E_5 = \left\{ v_{12i-10} v_{12i-5},\ 1 \le i \le \left\lfloor \frac{n-2}{8} \right\rfloor \right\}$$

$$E_6 = \left\{ v_{12i-4} v_{12i+1},\ 1 \le i \le \left\lfloor \frac{n-2}{8} \right\rfloor \right\}$$

$$E_7 = \left\{ v_{12i-7} v_{12i-2},\ 1 \le i \le \left\lfloor \frac{n-2}{8} \right\rfloor \right\}$$

$$E_8 = \left\{ v_{12i-1} v_{12i+4},\ 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor \right\}$$

$$E_9 = \{ v_1 v_4,\ v_{N-3} v_N \}$$

are the edges that are not in the Hamiltonian path.

Define the induced function $f^*: E(G) \rightarrow I$ (set of positive integers) as $f^*(v_i v_j) = f(v_i)\, f(v_j)$ for all $v_i v_j \in E$ and $v_i, v_j \in V$. The edge labeling for the edges along the Hamiltonian path are given by

$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+2), & i = 1,5,9,\dots N-5 \quad \text{if } n-2 = \text{multiple of } 8 \\ i\,(i+1), & i = N-1 \quad \text{if } n-2 = \text{multiple of } 8 \\ i\,(i+2), & i = 1,5,9,\dots N-3 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+1), & i = 2,6,10,\dots N-4 \quad \text{if } n-2 = \text{multiple of } 8 \\ i\,(i+1), & i = 2,6,10,\dots N-2 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$f^*(v_i v_{i+1}) = \begin{cases} (i-1)(i+1), & i = 3,7,11,\dots N-3 \quad \text{if } n-2 = \text{multiple of } 8 \\ (i-1)(i+1), & i = 3,7,11,\dots N-1 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+1), & i = 4,8,12,\dots N-2 \quad \text{if } n-2 = \text{multiple of } 8 \\ i\,(i+1), & i = 4,8,12,\dots N-4 \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$$

The edge labeling for the edges that are not in the Hamiltonian path are given by $f^*(v_{12i-10} v_{12i-5}) = (12i-9)(12i-6),\ 1 \le i \le \left\lfloor \frac{n-2}{8} \right\rfloor$

$f^*(v_{12i-4} v_{12i+1}) = (12i-4)(12i+1),\ 1 \le i \le \left\lfloor \frac{n-2}{8} \right\rfloor$

$f^*(v_{12i-7} v_{12i-2}) = (12i-7)(12i-1),\ 1 \le i \le \left\lfloor \frac{n-2}{8} \right\rfloor$

$f^*(v_{12i-1} v_{12i+4}) = (12i-2)(12i+4),\ 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor$

$f^*(v_1\ v_4) = 4$

$f^*(v_{N-3}v_N) = \begin{cases} N(N-4) \text{ if } n-2 = \text{multiple of } 8 \\ (N-3)N \quad \text{if } n-2 \neq \text{multiple of } 8 \end{cases}$

As seen in Subcase 1.1, it is observed from the above labeling that the edge labels within the edge sets and among the edge sets are distinct. Hence the graph under consideration admits strongly multiplicative labeling. An illustration is given in Fig 3b.

Case 2: If n is odd

$|V(G)| = N = \frac{3n-3}{2}$ and $|E(G)| = 2n - 3$. $v_1, v_2, \ldots\ldots v_N$ are the N vertices of G.

Define a bijection $f: V(G) \rightarrow \{1,2,3,\ldots..N\}$ as follows:

Subcase 2.1: If $n \equiv 1 \pmod 4$

$f(v_i) = \begin{cases} i, \quad i = 1,5,9,\ldots,N-3 \quad \text{if } n-1 = \text{multiple of } 8 \\ i, \quad i = 1,5,9,\ldots,N-1 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$f(v_i) = \begin{cases} i+1, \ i = 2,6,10,\ldots.N-2 \ \text{if } n-1 = \text{multiple of } 8 \\ i+1, \ i = 2,6,10,\ldots..N-4 \ \text{if } n-1 \neq \text{multiple of } 8 \\ i, \quad i = N \ \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$f(v_i) = \begin{cases} i-1, \quad i = 3,7,11,\ldots,..N-1 \quad \text{if } n-1 = \text{multiple of } 8 \\ i-1, \quad i = 3,7,11,\ldots,N-3 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$f(v_i) = \begin{cases} i, \quad i = 4,8,12,\ldots,N \quad \text{if } n-1 = \text{multiple of } 8 \\ i, \quad i = 4,8,12,\ldots,N-2 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

From the above labeling it is clear that f is a bijection from $V(G) \rightarrow \{1,2,3,\ldots..N\}$.

The edge set E of G is given by $E = E_1 \cup E_2 \cup E_3 \cup \ldots\ldots\ldots \cup E_9$ where

$E_1 = \begin{cases} v_iv_{i+1}, \quad i = 1,5,9,\ldots,N-3 \quad \text{if } n-1 = \text{multiple of } 8 \\ v_iv_{i+1}, \quad i = 1,5,9,\ldots,N-2 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$E_2 = \begin{cases} v_iv_{i+1}, \quad i = 2,6,10,\ldots,N-2 \quad \text{if } n-1 = \text{multiple of } 8 \\ v_iv_{i+1}, \quad i = 2,6,10,\ldots,N-1 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$E_3 = \begin{cases} v_iv_{i+1}, \quad i = 3,7,11,\ldots,N-1 \quad \text{if } n-1 = \text{multiple of } 8 \\ v_iv_{i+1}, \quad i = 3,7,11,\ldots,N-4 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$E_4 = \begin{cases} v_iv_{i+1}, \quad i = 4,8,12,\ldots,N-4 \quad \text{if } n-1 = \text{multiple of } 8 \\ v_iv_{i+1}, \quad i = 4,8,12,\ldots,N-3 \quad \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

are the edges along the Hamiltonian path and

$E_5 = \left\{ v_{12i-10}v_{12i-5}\,, 1 \le i \le \left\lfloor \frac{n-1}{8} \right\rfloor \right\}$

$E_6 = \left\{ v_{12i-4}v_{12i+1}\,, 1 \le i \le \left\lfloor \frac{n-3}{8} \right\rfloor \right\}$

$E_7 = \left\{ v_{12i-7}v_{12i-2}\,, 1 \le i \le \left\lfloor \frac{n-1}{8} \right\rfloor \right\}$

$E_8 = \left\{ v_{12i-1}v_{12i+4}\,, 1 \le i \le \left\lfloor \frac{n-3}{8} \right\rfloor \right\}$

$E_9 = \{v_1v_4\,, v_{N-4}v_N\}$ are the edges that are not in the Hamiltonian path.

Define the induced function $f^*: E(G) \rightarrow I$ (set of positive integers) as $f^*(v_iv_j) = f(v_i)\,f(v_j)$ for all $v_iv_j \in E$ and $v_i, v_j \in V$. The edge labeling for the edges along the Hamiltonian path are given by

$f^*(v_iv_{i+1}) = \begin{cases} i\,(i+2), \ i = 1,5,9,\ldots\ldots N-3 \ \text{if } n-1 = \text{multiple of } 8 \\ i\,(i+2), \ i = 1,5,9,\ldots\ldots N-2 \ \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$f^*(v_iv_{i+1}) = \begin{cases} i\,(i+1), \ i = 2,6,10,\ldots\ldots N-2 \ \text{if } n-1 = \text{multiple of } 8 \\ i\,(i+1), \ i = 2,6,10,\ldots\ldots N-1 \ \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$f^*(v_iv_{i+1}) = \begin{cases} (i-1)(i+1), \ i = 3,7,11,\ldots\ldots N-1 \ \text{if } n-1 = \text{multiple of } 8 \\ (i-1)(i+1), \ i = 3,7,11,\ldots\ldots N-4 \ \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

$f^*(v_iv_{i+1}) = \begin{cases} i\,(i+1), \ i = 4,8,12,\ldots\ldots N-4 \ \text{if } n-1 = \text{multiple of } 8 \\ i\,(i+1), \ i = 4,8,12,\ldots\ldots N-3 \ \text{if } n-1 \neq \text{multiple of } 8 \end{cases}$

The edge labeling for the edges that are not in the Hamiltonian path are given by $f^*(v_{12i-10}v_{12i-5}) = (12i-9)(12i-6),\ 1 \le i \le \left\lfloor\frac{n-1}{8}\right\rfloor$

$f^*(v_{12i-4}v_{12i+1}) = (12i-4)(12i+1),\ 1 \le i \le \left\lfloor\frac{n-3}{8}\right\rfloor$

$f^*(v_{12i-7}v_{12i-2}) = (12i-7)(12i-1),\ 1 \le i \le \left\lfloor\frac{n-1}{8}\right\rfloor$

$f^*(v_{12i-1}v_{12i+4}) = (12i-2)(12i+4),\ 1 \le i \le \left\lfloor\frac{n-3}{8}\right\rfloor$

$f^*(v_1\ v_4) = 4$

$$f^*(v_{N-4}v_N) = \begin{cases} N(N-4) \text{ if } n-1 = \text{multiple of } 8 \\ (N-1)\ (N-5) \quad \text{if } n-1 \ne \text{multiple of } 8 \end{cases}$$

As seen in Subcase 1.1, it is observed from the above labeling that the edge labels within the edge sets and among the edge sets are distinct. Hence the graph under consideration admits strongly multiplicative labeling. An illustration is given in Fig 4a.

Subcase 2.2: If $n \equiv 3 \pmod 4$

$$f(v_i) = \begin{cases} i, \quad i = 1,5,9,\dots,N-2 \quad \text{if } n-3 = \text{multiple of } 8 \\ i, \quad i = 1,5,9,\dots,N \qquad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i+1, \quad i = 2,6,10,\dots,N-1 \quad \text{if } n-3 = \text{multiple of } 8 \\ i+1, \quad i = 2,6,10,\dots.,N-3 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i-1, \quad i = 3,7,11,\dots,..N \quad \text{if } n-3 = \text{multiple of } 8 \\ i-1, \quad i = 3,7,11,\dots..,N-2 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$f(v_i) = \begin{cases} i, \quad i = 4,8,12,\dots,N-3 \quad \text{if } n-3 = \text{multiple of } 8 \\ i, \quad i = 4,8,12,\dots,N-1 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

From the above labeling it is clear that f is a bijection from $V(G) \rightarrow \{1,2,3,.....N\}$.The edge set E of G is given by $E = E_1 \cup E_2 \cup E_3 \cup \dots\dots\dots \cup E_9$ where

$$E_1 = \begin{cases} v_i v_{i+1}, \quad i = 1,5,9,\dots,N-2 \quad \text{if } n-3 = \text{multiple of } 8 \\ v_i v_{i+1}, \quad i = 1,5,9,\dots,N-4 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$E_2 = \begin{cases} v_i v_{i+1}, \quad i = 2,6,10,\dots,N-1 \quad \text{if } n-3 = \text{multiple of } 8 \\ v_i v_{i+1}, \quad i = 2,6,10,\dots,N-3 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$E_3 = \begin{cases} v_i v_{i+1}, \quad i = 3,7,11,\dots,N-4 \quad \text{if } n-3 = \text{multiple of } 8 \\ v_i v_{i+1}, \quad i = 3,7,11,\dots,N-2 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$E_4 = \begin{cases} v_i v_{i+1}, \quad i = 4,8,12,\dots,N-3 \quad \text{if } n-3 = \text{multiple of } 8 \\ v_i v_{i+1}, \quad i = 4,8,12,\dots,N-1 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

are the edges along the Hamiltonian path and

$$E_5 = \left\{v_{12i-10}v_{12i-5}, 1 \le i \le \left\lfloor\frac{n-1}{8}\right\rfloor\right\}$$

$$E_6 = \left\{v_{12i-4}v_{12i+1}, 1 \le i \le \left\lfloor\frac{n-3}{8}\right\rfloor\right\}$$

$$E_7 = \left\{v_{12i-7}v_{12i-2}, 1 \le i \le \left\lfloor\frac{n-3}{8}\right\rfloor\right\}$$

$$E_8 = \left\{v_{12i-1}v_{12i+4}, 1 \le i \le \left\lfloor\frac{n-4}{8}\right\rfloor\right\}$$

$E_9 = \{v_1v_4, v_{N-4}v_N\}$ are the edges that are not in the Hamiltonian path.

Define the induced function $f^*: E(G) \rightarrow I$ (set of positive integers) as $f^*(v_iv_j) = f(v_i)\,f(v_j)$ for all $v_iv_j \in E$ and $v_i, v_j \in V$. The edge labeling for the edges along the Hamiltonian path are given by

$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+2), \quad i = 1,5,9,\dots\dots N-2 \quad \text{if } n-3 = \text{multiple of } 8 \\ i\,(i+2), \quad i = 1,5,9,\dots\dots N-4 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+1), \quad i = 2,6,10,\dots\dots N-1 \quad \text{if } n-3 = \text{multiple of } 8 \\ i\,(i+1), \quad i = 2,6,10,\dots\dots N-3 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$f^*(v_i v_{i+1}) = \begin{cases} (i-1)(i+1), \quad i = 3,7,11,\dots\dots N-4 \quad \text{if } n-3 = \text{multiple of } 8 \\ (i-1)(i+1), \quad i = 3,7,11,\dots\dots N-2 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

$$f^*(v_i v_{i+1}) = \begin{cases} i\,(i+1), \quad i = 4,8,12,\dots\dots N-3 \quad \text{if } n-3 = \text{multiple of } 8 \\ i\,(i+1), \quad i = 4,8,12,\dots\dots N-1 \quad \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

The edge labeling for the edges that are not in the Hamiltonian path are given by $f^*(v_{12i-10}v_{12i-5}) = (12i-9)(12i-6),\ 1 \le i \le \left\lfloor \frac{n-1}{8} \right\rfloor$

$f^*(v_{12i-4}v_{12i+1}) = (12i-4)(12i+1),\ \ 1 \le i \le \left\lfloor \frac{n-3}{8} \right\rfloor$

$f^*(v_{12i-7}v_{12i-2}) = (12i-7)(12i-1),\ \ 1 \le i \le \left\lfloor \frac{n-3}{8} \right\rfloor$

$f^*(v_{12i-1}v_{12i+4}) = (12i-2)(12i+4),\ \ 1 \le i \le \left\lfloor \frac{n-4}{8} \right\rfloor$

$f^*(v_1\ v_4) = 4$

$$f^*(v_{N-4}v_N) = \begin{cases} (N-1)(N-5) & \text{if } n-3 = \text{multiple of } 8 \\ N\,(N-4) & \text{if } n-3 \ne \text{multiple of } 8 \end{cases}$$

As seen in Subcase 1.1, it is observed from the above labeling that the edge labels within the edge sets and among the edge sets are distinct. Hence the graph under consideration admits strongly multiplicative labeling. An illustration is given in Fig 4b.

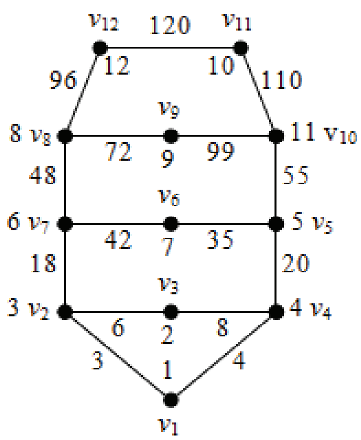

Fig. 4a Strongly multiplicative labeling of Cycle C_9 with Parallel P_3 chords

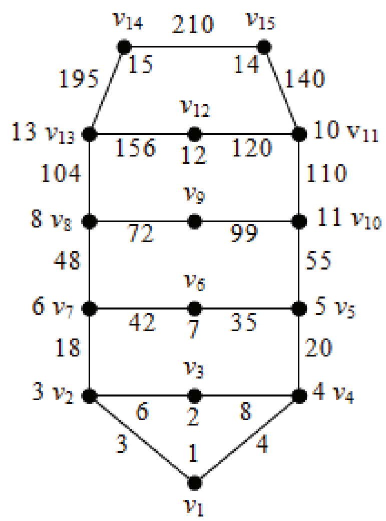

Fig. 4b Strongly multiplicative labeling of Cycle C_n with Parallel P_3 chords

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