ON CYCLIC COMMUTATIVITY NEAR – RINGS

G.Gopalakrishnamoorthy
Advisor, PSNL College of Education, Sattur -626 203
ggrmoorthy@gmail.com
R.Veega
Asst.prof.in Mathematics, Dr.G.R.D.College of Education, Coimbatore -641 402
veegavivek@gmail.com
Dr.M.Kamaraj
Associate Professor in Mahtematics, Government Arts and Science College,Sivakasi-626 123.
Kamaraj17366@gmail.com

Abstract: A right near – ring N is called weak commutative (Definition 9.4 Pilz [8]) if $xyz = xzy$ for every $x, y, z \in N$. A right near – ring N is called pseudo commutative (Definition 2.1 [9]) , if $xyz = zyx$ for all $x, y, z \in N$. A right near – ring N is called quasi weak commutative ([4]) , if $xyz = yxz$ for every $x, y, z \in N$. It is quite natural to investigate the properties of a right near – ring N satisfying $xyz = yzx$ for every $x, y, z \in N$. We call such a near –ring as cyclic commutative near – ring. We obtain some interesting results on cyclic commutative near – rings.

1. Introduction: Throughout this paper, N denotes a right near –ring (N, +, .) with atleast two elements. For any non – empty subset A of N, we denote $A - \{0\} = A^*$. The following definitions and results are well known.

1.1 Definition: An element $a \in N$ is said to be
1. Idempotent if $a^2 = a$.
2. Nilpotent, if there exists a positive integer k such that $a^k = 0$.

1.2 Result (Theorem 1.62 Pilz[7]): Each near – ring N is isomorphic toa subdirect product of subdirectly irreducible near – rings.

1.3 Definition: A near – ring is said to be zero symmetric if $ab = 0$ implies
$ba = 0$, where $a, b \in N$.

1.4 Result: If N is zero symmetric, then
1. Every left ideal A of N is an N – subgroup of N.
2. Every ideal I of N satisfies the condition $NIN \subseteq I$. That is, every ideal is an N – subgroup
3. $N^*I^*N^* \subseteq I^*$

1.5 Result: Let N be a near – ring. Then the following are true
1. If A is an ideal of N and B is any subset of N, then $(A: B) = \{ n \in N$ such that $nB \subseteq A\}$ is always a left ideal.
2. If A is an ideal of N and B is an N – sub group, then $(A: B)$ is an ideal. In particular if A and B are ideals of a zero – symmetric near – ring, then $(A: B)$ is an ideal.

1.6 Result:
1. Let N be a regular near – ring, $a \in N$ and $a = axa$, then ax, xa are idempotents and so the set of idempotent elements of N is non – empty.
2. $axN = aN$ and $Nxa = Na$.
3. N is S and S' near – rings.

1.7 Result (Lemma 4 Dheena [1]): Let N be a zero – symmetric reduced near – ring. For any $a, b \in N$ and for any idempotent element $e \in N$, $abe = aeb$.

1.8 Result (Gratzer [4] and Fain [3]): A near – ring N is sub – directly irreducible if and only if the intersection of all non – zero ideals of N is not zero.

1.9 Result (Gratzer [4]): Each simple near – ring is sub directly irreducible.

1.10 Result (Pilz[7]): A non – zero symmetric near – ring N has IFP if and only if $(0 : S)$ is an ideal for any subset S of N.

1.11 Result (Oswald [6]): An N – subgroup A of N is essential if $A \cap B = \{0\}$, where B is any subgroup of N, implies $B = \{0\}$.

1.12 Definition: A near – ring N is said to be reduced if N has no non – zero nilpotent elements.

1.13 Definition: A near – ring N is said to be an integral near –ring, if N has no non – zero divisiors.

1.14 Lemma: Let N be a near – ring. If for all $a \in N, a^2 = 0 \implies a = 0$, then N has no non – zero nilpotent elements.

1.15 Definition: Let N be anear – ring . N is said to satisfy intersection of factors property (IFP) if $ab = 0 \implies anb = 0$ for all $n \in N$, where $a, b \in N$.

1.16 Definition:

1. An ideal I of N is called a prime ideal if for all ideals A, B of N , AB is subset of I $\implies$ A is subset of I of B is subset of I.
2. I is called a semi – prime ideal if for all ideals A of N, A^2 is subset of I implies A is subset of I.
3. I is called a completely semi – prime ideal, if for any $x \in N$, $x^2 \in I \implies x \in I$.
4. A completely prime ideal, if for any $x , y \in N$, $x, y \in I \implies x \in I$ or $y \in I$.
5. N is said to have strong IFP, if for all ideals I of N, $a, b \in I$ implies $an\,b \in I$ for all $n \in N$.

1.17 Result: Let N be a Pseudo commutative near – ring. Then every idempotent element is central.

2. Main Results

2.1 Theorem: Every cyclic commutative near –ring is zero symmetric.

Proof: Let N be a cyclic commutative right Near – ring.

Let $a \in N$ be any element.

Now $a.0 = a.(0.0)$
$= (0.0)a$ (Cyclic Commutative)
$= 0(0.a)$
$= 0.0$
$= 0$

Thus N is zero symmetric.

2.2 Theorem: Let N be a right near – ring.

(i) If N is both cyclic and weak – commutative, then N is quasi – weak commutative.

(ii) If N is both cyclic and pseudo commutative, then N is weak commutative.

(iii) If N is both Cyclic and quasi – weak commutative, then N is pseudo commutative.

Proof:

(i) Let N be both cyclic and weak – commutative near – ring.

For all $x, y, z \in N$,
$xyz = yzx$ (cyclic commutative)
$= yxz$ (weak commutative)

This implies N is quasi weak commutative.

(ii) Let N be both cyclic and pseudo commutative .

For all $x, y, z \in N$,

$xyz = yzx$ (cyclic commutative)

$= xzy$ (pseudo commutative)

This implies N is weak commutative.

(iii) Let N be both cyclic and quasi weak commutative .

For all $x, y, z \in N$,

$xyz = yzx$ (cyclic commutative)

$= zyx$

This implies N is pseudo commutative.

2.3 Theorem: Homomorphic image of a cyclic commutative near – ring is cyclic commutative.

Proof: Let N be a cyclic commutative right near – ring and $f : N \rightarrow M$ be an endomorphism of near – rings N and M.

For all $x, y, z \in N$,

$$f(x)f(y)f(z) = f(xyz)$$
$$= f(yzx)$$
$$= f(y)f(z)f(x)$$

So M is cyclic Commutative.

2.4 Corollary: Let N be a cyclic commutative near – ring .If I is any ideal of N, then $N/_I$ is also cyclic commutative.

2.5 Theorem: Every cyclic commutative near ring N is isomorphic to a sub direct product of sub directly irreducible cyclic commutative near – rings.

Proof: By result 1.2, N is isomorphic to a sub – direct product of sub – directly irreducible near rings N_α each of which is homomorphic image of N under the projection map $\pi_\alpha : N \rightarrow N_\alpha$.

The desired result follows from Theorem 2.3.

2.6 Theorem: Any weak Commutative near –ring with left identity is cyclic commutative.

Proof: Let $e \in N$ be a left identity

For all $a, b, c \in N$, $abc = e(abc) = (eab)c$

$= (eba)c$ (weak commutative)

$= bac$

Therefore N is cyclic commutative.

2.7 Theorem: Any pseudo commutative near – ring with left identity is cyclic commutative.

Proof: Let $e \in N$ be a left identity

For all $a, b, c \in N$, $abc = e(abc)$

$= (eab)c$

$= (bae)c$ (pseudo commutative)

$= b(aec)$

$= b(cea)$ (pseudo commutative)

$= (bc)(ea)$

$= bca$

Therefore N is cyclic commutative.

2.8 Theorem: Let N be a regular cyclic commutative near – ring. Then

(i) $A = \sqrt{A}$ for every N – subgroup A of N.

(ii) N is reduced.

(iii) N has (IFP)

Proof: Let N be a regular cyclic commutative near –ring.

Since N is regular, for every $a \in N$, there exists $b \in N$ such that

$a = aba$

$= baa$ (cyclic commutative)

$a = b\,a^2$

(i) Let A be a N – subgroup of N and let $a \in \sqrt{A}$. Then $a^k \in A$ for some positive integer k. If $a \in N$ there exists $b \in N$ such that $a = ba^2$

So,

$$
\begin{aligned}
a &= b(a)a \\
&= b(ba^2)a \\
&= b^2a^3 \\
&= b^2(a)a^2 \\
&= b^2(ba^2)a^2 \\
&= b^3a^4 \\
&\;\;\vdots \\
&= b^{(k-1)}a^k \in N\,A
\end{aligned}
$$

which is a subset of A. So $\sqrt{A}$ is a subset of A. Always A is a subset of $\sqrt{A}$.

Hence $A = \sqrt{A}$.

This completes the proof of of (i)

(ii) If $a^2 = 0$, then by (i) $a = ba^2 = b.0 = 0$

This completes the proof (ii)

(iii) Let $a, b \in N$ such that $ab = 0$

Then $(ba)^2 = (ba)(ba) = b(ab)a$

$= b(0a) = b.0 = 0$

So, by (ii) $ba = 0$

Thus $ab = 0$ implies $ba = 0$.

Now for any $n \in N$,

$$
\begin{aligned}
(anb)^2 &= (anb)(anb) \\
&= (an)(ba)(nb) \\
&= (an)0(nb) \\
&= (an)0 \\
&= 0
\end{aligned}
$$

Again by (ii) $anb = 0$

This proves (iii) .

2.9 Theorem: Let N be a regular cyclic commutative near – ring. Then every N subgroup id an ideal and $N = Na = Na^2 = aN = aNa$ for all $a \in N$.

Proof: Let $a \in N$. Since N is regular there exists $b \in N$ such that $a = aba$.

Then by Result 1.6, (ba) is idempotent.

Let $ba = e$. Then $Ne = Nba = Na$ (by Result 1.6) ...... (1)

Let $S = \{ n - ne \,/ n \in N\}$.

We claim that, $(0{:}S) = Ne$

Now $(n - ne)e = ne - ne^2 = ne - ne = 0$ for all $n \in N$.

So, $(n - ne)Ne = 0$ by (iii) of Theorem 2.8.

This implies $Ne \subseteq (0{:}S)$

Now, let $y \in (0{:}S)$.

Then $sy = 0$ for all $s \in S$ ...... (2)

Since N is regular, $y = yxy$ for some $x \in N$.

Since $yx - (yx)e \in S$, $(yx - (yx)e)y = 0$ (by (2))

That is, $yxy - yxey = 0$

$y - (yx)(eey) = 0$ (e is idempotent)

$y - (yx)(eye) = 0$ (cyclic commutativity)

$y - y\,(xey)\,e = 0$

$y - y\,(eyx)e = 0$ (cyclic commutativity)

$y - (yey)xe = 0$

$y - (eyy)xe = 0$ (cyclic commutativity)

$y - e\,(yyx)e = 0$

$y - e(yxy)e = 0$ (cyclic commutativity)

$y - eye = 0$

$y - yee = 0$ (cyclic commutativity)

$y - ye = 0$

That is, $y = ye \in Ne$

It follows that $(0{:}\,S) \subseteq Ne$

Thus $(0{:}\,S) = Ne$.

By Result 1.10, Na is an ideal of N.

Now, if M is any subgroup of N, then $M = \sum_{a \in N} Na$.

Thus M becomes an ideal of N.

Since N is regular, for every $a \in N$, there exists $b \in N$ such that $a = aba$.

Now $a = aba = baa$ (cyclic commutativity)

$= ba^2 \in Na^2$

So, $N \subseteq Na^2$

Now $Na \subseteq N \subset Na^2 \subseteq (Na)a \subseteq Na \subseteq N$

So, $Na = Na^2 = N$.

Now we shall prove that $Na^2 = aNa$

Let $x \in Na^2$.

Then $x = na^2$

$= naa$

$= aan$ (cyclic commutativity)

$= ana$ (cyclic commutativity)

So, $x \in aNa$

That is, $Na^2 \subseteq aNa$.

Let $a \in aNa$. Then $y = ana$ for some $n \in N$

$= naa$ (cylic commutativity)

$= na^2 \in Na^2$

That is, $aNa \subseteq Na^2$

Thus $aNa = Na^2$.

Next , we claim that $aN = aNa$.

Since Na is an ideal, for every $a \in N$, $(Na)N \subseteq Na$

Also for every $n \in N$, $an = (aba)n = a(ban) \in a(NaN) \subseteq aNa$

Thus $aN \subseteq aNa$

Obeviousely $aNa \subseteq aN$

Hence $aN = aNa$.

Thus $N = Na = Na^2 = aN = aNa$ for all $a \in N$.

2.10 Definition: A near – ring N is said to have property P_4 , if $ab \in I \Longrightarrow ba \in I$ where I is any ideal of N.

2.11 Theorem: Let N be a regular cyclic commutative near – ring .
Then (i) every ideal of N is Completely semi prime
(ii) N has property P_4

Proof:

(i) Let I be any ideal of N.

Let $a^2 \in I$. Since N is regular , there exists $b \in N$ such that $a = aba$.

Now a = aba = baa (cyclic commutativity)

$= ba^2 \in NI \subseteq I$

This implies $a \in I$

So, every ideal of N is completely semi – prime.

(ii) Let $ab \in I$

Then $(ba)^2 = (ba)(ba) = b(ab)a \in NIN \subseteq N$ (by Result 1.4)
Then by (i) $ba \in I$.
Thus $ab \in I \Rightarrow ba \in I$
Thus N has property P_4 .

2.12 Theorem: Let N be a regular commutative near – ring. For every ideal I of N, $(I{:}\,S)$ is an ideal of N, where S is any subset of N.

Proof: Let I be an ideal of N and S be any subset of N.
By Result 1.5 (ii), $(I : S) = \{n \in N/ns \subseteq I\}$ is a left ideal of N. Let $s \in S$ and $a \in (I{:}\,S)$, then $as \in I$.
Since N has property P_4 , $sa \in I$.
Then for any $n \in N$, $(sa)n \in I$
That is, $s(an) \in I$
So, $(an)s \in I$ (Since N has property P_4)
That is, $an \in (I{:}\,S)$ for any $n \in N$ and have $(I{:}\,S)$ is a right ideal. Consequently $(I{:}\,S)$ is an ideal.

2.13 Theorem: Let N be a regular cyclic commutative near – ring. For any ideal I of I and $x_1, x_2, \dots, x_n \in N$, if $x_1.x_2.x_3 \dots .x_n \in I$, then $< x_1 >< x_2 >< x_3 > \cdots < x_n > \subseteq I$.

Proof: Let $x_1.x_2.x_3 \dots .x_n \in I$

$\Rightarrow x_1 \in (I : x_2.x_3 \dots .x_n)$
$\Rightarrow < x_1 > \subseteq (I : x_2.x_3 \dots .x_n)$
$\Rightarrow < x_1 > x_2.x_3 \dots .x_n \subseteq I$
$\Rightarrow x_2 \in (I : x_3 \dots .x_n) < x_1 >$
$\Rightarrow < x_2 > \subseteq (I : x_3.x_4 \dots .x_n) < x_1 >$
$\Rightarrow < x_2 > x_3x_4 \dots x_n \subseteq I$.
Continuing like this we get $< x_1 >< x_2 >< x_3 > \cdots < x_n > \subseteq I$.

2.14 Theorem: Let N be a cyclic commutative ring. Then
(i) N has strong IFP
(ii) N is semi – prime near – ring

Proof:

(i) Let I be a ideal of N. Since N is zero symmetric, $NI \subseteq I$. By Theorem 2.9, $aN = Na^2$. Hence $am = ma^2$ for some $n, m \in N$.
Hence if $ab \in I$, then for every $n \in N$, $anb = ma^2b = ma(ab) \in NI \subseteq I$
That is, $ab \in I \Rightarrow anb \in I$ for all $n \in N$
This proves N has strong IFP.

(ii) Let M be a N subgroup of N. Then M is an ideal by Theorem 2.9. Let I be any ideal of N such that $I^2 \subseteq M$. Then by result 1.4, $NI \subseteq I$.
If $a \in I$, then $a = aba$ for some $b \in N$ (Since N is regular)
This implies $a = aba \in I(NI) \subseteq I^2 \subseteq M$. So, any N – subgroup M of N is a semi – prime ideal.
In particular {0} is a semi – prime ideal and hence N is a semi – prime near – rinmg.

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