WIENER INDEX OF GENERALIZED BOOKS

L. Nirmala Rani
Department of Mathematics, Karunya University, Coimbatore, India
nirmala_ranisamy@yahoo.co.in
Jennifer Rajkumari.K
Electrical Engineer, Palo Alto, California, USA
S. Maria Jesu Raja
Department of Mathematics, Loyola College, Chennai, India

Abstract: Grid embedding plays an important role in computer architecture. VLSI Layout Problem, Crossing Number Problem, Graph Drawing and Edge Embedding Problem are all a part of grid embedding. In interconnection networks, so far the hypercube has been the focus of many researchers due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [1].Hypercubes are known to simulate other structures such as grids and binary trees [2,3].But the degree of vertices of hypercubes increases with the size, preventing hardware expendability. This is not the case for the grid or the torus because of its vertex transitive extension. This advantage gives renewed interest in this ancient topology namely the grid network, which was the first one proposed for parallel computers and which, in spite of a rather large diameter, has gained popularity. Generalized books are extensions of the grid networks [4]. In this paper we find the Wiener index of generalized books.

Keywords: Undirected Graph, Shortest Distances, Wiener Index, Generalized Books.

Introduction: A topological index is a real number related to a graph. It is a structural invariant and is preserved by every graph automorphism. Several topological indices have been defined and many of them have found applications as means to model chemical, pharmaceutical and other properties of molecules [1, 11]. Usage of topological indices in chemistry began in 1947 when chemist Harold Wiener developed the most widely known topological descriptor, the Wiener index and used it to determine physical properties of types of alkanes known as paraffin's. It is named after his inventor Harry Wiener (1924-1998). There are many topological indices such as Hyper-Wiener index, Eata index, PI index etc.

Wiener Index: In theoretical Computer Science, Wiener index is considered as one of the basic descriptors offixed interconnection networks as it provides the average distance between any two nodes of the network. Wiener index can be calculated in a number of ways.

a) Multiply the number of vertices on each side of an edge and add all such contributions.
b) (2)It is also equal to half the sum of the off-diagonal elements D_i of the distance matrix where D_i represents the length of the shortest path between any two vertices in a graph.
c) The Wiener index W of a graph G is equal tohalf the sum of the off-diagonal elements of the distance matrix D : W=$\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} D_{ij}$ where D_{ij} represents the length of a shortest path between vertices i and j in$G$.
d) Consequently, Wiener index based on all pairs shortest path problem is defined as half the sum of all shortest paths between unordered pairs of vertices of a graph [6,7,8,9,10,11,12,13].

Definition 1.2: Let G be a connected undirected graph with vertex set $V(G)$ and edge set $E(G)$.Let $d\ (v_i, v_j)$be the shortest distance between the vertices v_i and v_j. The Wiener index has been defined for Gas half the sum of all shortest distances $d(v_i, v_j)$ between the unordered pair of vertices of G
i.e.$D(G) = \frac{1}{2}\ d(v_i, v_j)$ where $v_i, v_j \epsilon V(G)$

In this paper an elegant technique has been used without using distance matrix to compute the Wiener index of a generalized book. To calculate the Wiener index of generalized books we use the following results based on edge-cut techniques.

Edge-Cut Partition Lemma: Let G be a graph on n vertices. Let$\{S_1, S_2, S_3, \ldots, S_m\}$be a partition of $E(G)$ such that each S_i is an edge cut of G and the removal of edges of $S_i$leaves G into two components $G_i$and G'_i. Also each S_i satisfies the following conditions

a) For any two verticesu, v in G_ia shortest path between u and v has no edges in S_i.
b) For any two verticesu, vin G'_i, a shortest path between u and v has no edges in S_i
c) For any two vertices$u$in G_i and v in G'_i, a shortest path between u and v has exactly one edge inS_i. Then the congestion on $S_i$is given by $c(S_i) = |V(G_i)|(n - |V(G_i)|)$ and Wiener index of $G$is given by $W(G) = \sum_{i=1}^{m} |V(G_i)|(n - |V(G_i)|)$[5].

Edge-Cut k-Partition Lemma: Let$G$ be a graph on n vertices. Let $E^k(G)$ denote a collection of edges of G with each edge in G repeated exactly k times. Let$\{S_1, S_2, S_3, \ldots, S_m\}$be a partition of $E(G)$ such that each S_i is an edge cut of G and the removal of edges of S_i leaves G into two components $G_i$and G'_i. Also each S_i satisfies the following conditions

a) For any two vertices u, v in G_i, a shortest path between u and v has no edges inS_i.
b) For any two verticesu, v in G'_i, a shortest path between$u$and $v$has no edges inS_i
c) For any two vertices u in G_i and v in G'_i, a shortest path between u and v has exactly one edge inS_i. Then the congestion on $S_i$is given by $c(S_i) = |V(G_i)|(n - |V(G_i)|)$ and Wiener index of G is given by $W(G) = \frac{1}{k}\sum_{i=1}^{m} |V(G_i)|(n - |V(G_i)|)$[5].

Generalized Book Type 1:

Definition 2.1: Let $M[m \times n]$ be an $m\times n$ mesh with m rows and n columns. A graph which is obtained from l copies of M, say$\{M_1, M_2, M_3, \ldots, M_l\}$ by merging each vertex of the first column of $M_1$with the corresponding vertex of the first column of M_i, for all$i = 2, 3, \ldots, l$ is called a generalized book of Type 1 and is denoted by$GB[m,n,l]$. See Figure 1.

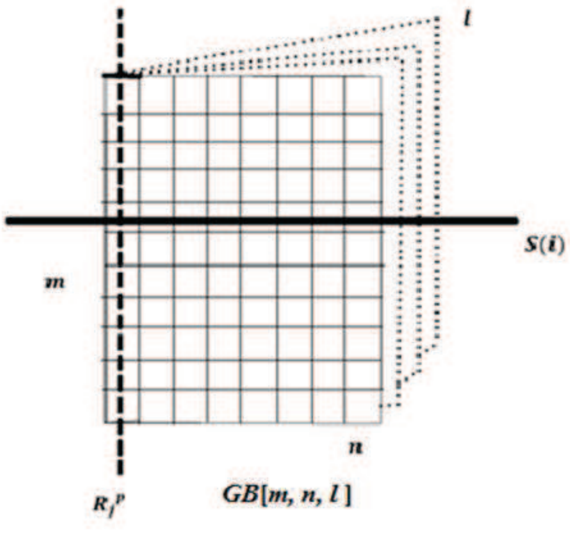

Figure 1: Generalized Book $GB[n, m, l]$ of type 1

Remark2.2: $GB[m, n, l]$ has $k = m(l\,n - l + 1)$ vertices.

Theorem 2.3: Let G be the generalized book $GB[m, n, l]$. Then the Wiener index of G is given by

$$W\,(G) = \sum_{i=1}^{m-1}(k - i(ln\ - l + 1)\,)i(ln\ - l + 1)\,) + \sum_{j=1}^{n-1}(k - jm)jm$$

Proof: Let *(p, i, j)* denote the vertex in the i^{th} row and j^{th} column of the p^{th} copy of M, $1 \le i \le m, 2 \le j \le n, 1 \le p \le l$. For $1 \le i \le m$, $1 \le p \le l$, let *(i,1)* denote the i^{th} element in the merged first column of every M_p. For $1 \le i \le m-1$, define $S_p\ (i) = \bigcup_{j=2}^{n}((p,i,j)\ ,(p,i+1,j))$ and $S(i) = \{(i,1),(i+1,1)\} \cup \bigcup_{p=1}^{l} S_p\ (i)$. For $1 \le p \le l$, $2 \le j \le n-1$ define $R_1^p = \bigcup_{i=1}^{m}\{((i,1),(p,i,2))\}$ and $R_j^p = \bigcup_{i=1}^{m}\{(\ (p,i,j),(p,i,j+1))\ \}$. Now S(i) cuts$G$ into two components GS(i) and GS'(i) with one component having exactly $i(ln\ - l + 1)$ vertices. Further R_j^p cuts$G$ into two components one with exactly jm vertices.Now S=S(i)$\cup$ R_j^p is a partition of the edge set of G satisfying the conditions of the Edge-Cut-Partition Lemma. Let $c(S)$ denote the congestion on$S$. Then the Wiener index $W(G)$ of g is given by

$$W(G) = c(S) = \sum_{i=1}^{m-1} c(\,S_p\ (i)) + \sum_{j=1}^{n-1} c(R_j^p)$$

$$= \sum_{i=1}^{m-1}(k - i(ln\ - l + 1)\,)i(ln\ - l + 1)\,) + \sum_{j=1}^{n-1}(k - jm)jm$$

Illustration: $m = 5, n = 4, l = 2$; $W(GB[5,4,2]) = 2380$.

Table 1: Wiener Index of Generalized Book Type 1

S.No	Cuts	Congestion on Cuts	Total
1	S_1	(28)(7)	195
2	S_2	(21)(14)	294
3	S_3	(14)(21)	294
4	S_4	(7)(28)	196
5	R_1	(5)(30)(2)	300
6	R_2	(10)(25)(2)	500
7	R_3	(15)(20)(2)	600
			2380

Generalized Book Type 2:

Definition 3.1: Let $M[m \times n]$ be an $m \times n$ mesh with m rows and n columns. A graph which is obtained from l copies of M, say$\{M_1,\ M_2\ ,M_3\ ,\ldots,\ M_l\}$ by joining each vertex of the first column of $M_1$with the corresponding vertex of the first column ofM_i, for all i = 2, 3, . . . , l is called a generalized book of Type 2 and is denoted by$GB^*[m,n,l]$. See Figure 2.

Remark 3.2: GB*$[m,n,l]$ has lmn vertices.

Theorem 3.3: Let$G$be the generalized book GB*$[m,n,l]$. Then the Wiener index of G is given by

$$W(G) = \sum_{i=1}^{m-1}(lmn - in)in + \sum_{j=1}^{n-1}(lmn - jm)jm\ + (l-1)(\,lmn - mn)\,mn$$

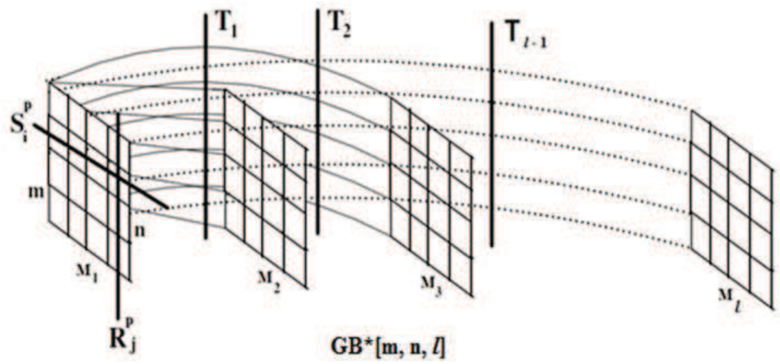

Figure 2: Generalized Book $GB^*[n, m, l]$ of type 2

Proof: Let (p, i, j) denote the vertex in the i^{th} row and j^{th} column of the p^{th} copy of M, $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq p \leq l$. For $1 \leq i \leq m-1$ define $S_i^p = \cup_{j=1}^{n}((p,i,j), (p,i+1,j))$. For $1 \leq p \leq l$, $1 \leq j \leq n-1$ define $R_j^p = \cup_{i=1}^{m}\{((p,i,j), (p,i,j+1))\}$. For $1 \leq i \leq m$, $1 \leq p \leq l-1$, let T_p denote the set of edges $((p,i,1), (p+1,i,1)$ joining vertices in column 1 of M_p with the Corresponding vertices in column 1 of M_{p+1}. Now S_i^p cuts $GB^*[n, m, l]$ into two components GS^*(i) and GS'^*(i) with one component having exactly in vertices. Similarly R_j^p cuts G into two components, one with exactly jm vertices. Further T_p divides G into two components, one of the components with exactly mn vertices.Now $S = S_i^p \cup R_j^p \cup T_p$ is a partition of the edge set of G satisfying the conditions of the Edge-Cut-Partition Lemma. Let $c(S)$ denote the congestion on S. Then the Wiener index $W(G)$ of G is given by

$$W(G) = c(S) = \sum_{i=1}^{m-1} c(S_i^p) + \sum_{j=1}^{n-1} c(R_j^p) + \sum_{p=1}^{l-1} c(T_p)$$

$$= \sum_{i=1}^{m-1} (\boldsymbol{l}mn - in)in + \sum_{j=1}^{n-1} (\boldsymbol{l}mn - jm)jm + (\boldsymbol{l} - 1)(lmn - mn)mn$$

Illustration: $m = 3, n = 4, l = 3$; $W(GB\ [m;\ n;\ l])$ =1450.

Table 2: Wiener Index of Generalized Book Type 2

	Cuts	Congestion on Cuts	Total
1	S_1	4)(32)	128
2	S_2	(8)(28)	224
3	R_1	(3)(33)	99
4	R_2	(6)(30)	180
5	R_3	(9)(27)	243
6	K_1	12)(24)	288
7	K_2	(24)(12)	288
			1450

Conclusion: In this paper we have found the Wiener index of generalized books of type 1 and type 2. It would be quite interesting to find Wiener index of other variations of mesh network namely chain of generalized books.

References:

1. Y. Saad, M.H. Schultz; Topological properties of hypercubes; IEEE Trans. Comput. 37 (7) 867-872, 1988.
2. P. Manuel, I. Rajasingh, B. Rajan, and H. Mercy; Exact wirelength of hypercube on a grid, Discrete Appl. Math. 157(7) pp. 1486-1495, 2009.
3. W.K. Chen and M.F.M. Stallmann; on embedding binary trees into hypercubes; Journal on Parallel and Distributed Computing, Vol. 24,132-138, 1995.
4. R.SundaraRajan ,IndraRajasingh, N.Parthiban,T.M.Rajalaxmi; A linear time algorithm for embedding hypercube into cylinder and torus;Theoretical Computer Science 542 108-115, 2014.
5. Paul Manuel, IndraRajasingh, BharatiRajanandR.SundaraRajan, A New Approach to Compute Weiner Index, Journal of Comp. andtheoriticalNanosci.,10,1-7 , 2013.
6. I. Gutman, B. Furtula, M. Petrovi, Terminal Wiener index, J. Math. Chem. 46 522-531. 2009.
7. I. Gutman, Y. N. Yeh, S. L. Lee, Y. L. Luo, Some recent results in the theory of the Wiener number, Indian J. Chem. 32A ,651-661,1993.
8. A. Graovac, T. Pisanski, On the Wiener index of a graph, J. Math. Chem.8 ,53-62, 1991.
9. I. Gutman, A formula for the Wiener number of trees and its extension to graphs containing cycles, Graph Theory Notes New York 27 ,9-15, 1994.
10. A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, Acta Appl. Math.66, 211-249, 2001.
11. M. Randi, On generalization of Wiener index for cyclic structures, ActaChim. Slov. 49, 483-496, 2002.
12. P. V. Khadikar, N. V. Deshpande, P. P. Kale, A. A. Dobrynin, I. Gutman, G. Dmtor, The Szeged index and an analogy with the Wiener index,J. Chem. Inf. Comput. Sci. 35, 547-550,1995.
13. B.Mohar, T. Pisanski; How to compute the Weiner index of a Graph; J.Math.Chem.2, 267-27, 1988.

***