A NEW NOTION OF CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: The aim of this paper is to introduce a new class of closed sets namely β*- closed sets and discuss its properties with already existing other sets. Additionally, we define the complement of β*- closed sets, and we find the basic properties and characterizations of β*- open sets in topological spaces.

Keywords: β *- Open, β *- Closed.

Introduction: In general topology repeated application of interior and closure operators give rise to several different new classes of sets. Some of them are generalized form of open sets. These classes are found to have applications not only in mathematics but even in diverse fields outside the realm of mathematics ([7] , [8], [18]). The most well known notion and inspiring sources are the notions of β–open (semi–preopen) sets by Abd–El–Monsef et al. [13] (by Andrijevic ([1], [2])). Due to this, investigation of these sets have gained momentum in recent days. By originating the concept of generalized closed (g–closed) sets, Levine [9] provided an umbrella for the researchers working in the field of generalized closed sets. Levine [9] used the closure operator and the openness of the superset in the definition of g–closed sets.

In [17] Robert et al. originated the concept of semi*–closed sets by using the closure operator Cl* due to Dunham [5]. They investigated many fundamental properties of semi*–closed sets. This class of set lies between closed sets and semi–closed sets. They also established semi*–closure of semi*- closed set. Missier [15] devised and studied the new notion of sets called α*–open sets and α*–closed sets and discussed the relationship of α*–open sets and α*–closed sets with some other sets. Selvi et al. [20] defined and investigated a new class of sets called pre*–closed sets by using the generalized closure operator Cl* due to Dunham [5].

In this paper a new notion of generalized closed sets namely β *- closed sets has been devised. A brief synopsis of the paper is as follows: The main objective of this paper is to introduce and study β *- closed sets, which is the generalization of β–closed sets by using the generalized closure operator Cl*. This class of sets are the generalization of β *- closed sets, pre*–closed sets and semi*–closed sets. This paper is organized as follows, section 1, gives basic notions which underpin our work. In section 2, we have define β *- closed sets and discuss their characterization and basic properties and its relationships with already existing generalized closed sets.

Preliminaries: Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ), Cl(A) and Int

(A) denote the closure and the interior of A respectively. Here we recall the following known definitions and properties.

Definition 2.1: Let (X, τ) be a topological space. A subset A of the space X is said to be pre- open [14] if A ⊆ Int (Cl(A)) and pre - closed if Cl(Int (A)) ⊆ A.

Definition 2.2.: Let (X, τ) be a topological space. A subset A of the space X is said to be semi–open [11] if A ⊆ Cl(Int (A)) and semi–closed if Int (Cl(A)) ⊆ A.

Definition 2.3: Let (X, τ) be a topological space. A subset A of the space X is said to be α–open [14] if A ⊆ Int (Cl(Int (A)) and α–closed if Cl(Int (Cl(A)) ⊆ A.

Definition 2.4: Let (X, τ) be a topological space. A subset A of the space X is said to be β–open [13] if A ⊆ Cl(Int (Cl(A))) and β–closed if Int (Cl(Int (A))) ⊆ A.

Definition 2.5: Let (X, τ) be a topological space. A subset A of the space X is said to be generalized closed (briefly g–closed)[9] if Cl(A) ⊆ U whenever A ⊆ U and U is open in X. Generalized open(briefly g–open) if X \ A is g–closed.

Definition 2.6: Let (X, τ) be a topological space. A subset A of the space X is said to be pre*–closed set [20] if Cl*(Int (A)) ⊆ A and pre*–open set if A ⊆ Int*(Cl(A)).

Definition 2.7: Let (X, τ) be a topological space. A subset A of the space X is said to be semi*–closed set [17] if Int * (Cl(A)) ⊆ A and semi*–open set [8] if A ⊆ Cl*(Int (A)).

β ***-Closed Set:** In this section we introduce β *- closed Sets and investigate some of their basic properties.

Definition 3.1: A subset A of a topological space (X, τ) is called β *- closed Set if Int(Cl*(Int(A))) ⊆ A.
Let β *C(X) denotes the collection of all β *- closed Sets in X.

Example 3.2: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a, b, c},{a, b}}, then (X,τ) be a topological space.
τ^c = {ϕ, X,{d},{c, d}},
gC(X) = {ϕ, X,{d},{a, d},{ b, d},{c, d},{a, b, d},{a, c, d},{b, c, d}},
gO(X) = {X, ϕ, {a},{b},{c},{a, b}, {a, c}, {b, c}, {a, b, c}}
β *C(X) = {X, ϕ,{a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, (b, d}, {c, d},{a, b, d}, {a, c, d},{b, c, d}}.

Definition 3.3: A subset A of a topological space (X, τ) is called β *- open Set if X \ A is β *- closed Set.
Let β *O(X) denotes the collection of all β *- open Sets in X.

Example 3.4: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a},{b},{c},{a, b},{a, c}, {b, c}, {a, b, c}} and τ^c = {ϕ, X,{d},{a, d}, {b, d},{c, d},{a, b, d}, {a, c, d},{b, c, d}}, then
gC(X) = {ϕ, X,{d},{a, d},{ b, d},{c, d},{a, b, c},{a, b, d},{a, c, d},{b, c, d}},
gO(X) = {X, ϕ, {a},{b},{c},{d},{a, b}, {a, c}, {b, c}, {a, b, c}}
β *O(X) = {X, ϕ,{a}, {b}, {c},{a, b},{a, c}, {a, d},{b, c},(b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d},{b, c, d}}

Theorem 3.5: Let (X,τ) be a topological space then every open set of (X,τ) is β *- open.
Proof: Let A be any open set in X . Every open set is β - open then , we have A ⊆ Cl (Int(Cl (A))) ⊆ Cl (Int *(Cl (A))).Hence A is β *- open.

Remark 3.6: The converse of the above theorem need not be true.

Example 3.7: Let X = {a, b, c } be any set and τ = {X, ϕ,{a, b}}, then (X,τ) be a topological space. τ^c= {ϕ, X,{c}}, then β *O(X) = {X, ϕ,{a}, {b}, {a, b},{a, c}, {b, c}}. Clearly the sets {a}, {b}, {a, c}, {b, c} are β *-open but not open.

Theorem 3.8: Let (X,τ) be a topological space then every β - open set of (X,τ) is β *- open.

Proof: Let A be any β - open set in X . Then A ⊆ Cl (Int(Cl (A))) ⊆ Cl (Int *(Cl (A))). Hence A is β *-open.

Remark 3.9: The converse of the above theorem need not be true.

Example 3.10: Let X = {a, b, c} be any set and τ = {X, ϕ,{a}}, then (X,τ) be a topological space. τ^c = {ϕ, X,{b, c}} then β *O(X) = {X, ϕ,{a}, {b}, {c},{a, b},{a, c}, {b, c}} and β - open = { X, ϕ,{a},{a, b},{a, c}}. But {b}, {c}, {b, c} are β *- open but not β - open.

Theorem 3.11: Let (X,τ) be a topological space then every g-open set of (X,τ) is β *- open.

Proof: Let A be any g- open set in X . Then Int*(A) = A and A ⊆ Cl (A). Int* A ⊆ Int*(Cl (A))) ⊆ Cl (Int *(Cl (A))). Hence A is β *- open.

Remark 3.12: The converse of the above theorem need not be true.

Example 3.13: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a}, {b}, {a, b}, {a, b, c}}, then (X,τ) be a topological space. τ^c = {ϕ, X,{d},{c, d},{a, c, d},{b, c, d}}, then β *O(X) = {X, ϕ,{a}, {b}, {c},{a, b}, {a, c}, {a, d},{b, c},(b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d},{b, c, d}} and gO(X) = { X, ϕ,{a}, {b},{c}, {a, b}, { a, c}, {b, c}, {a, b, c}} . But {a, d},{b, d},{c, d},{a, b, d}, {a, c, d},{b, c, d} are β *- open but not g - open.

Theorem 3.14: Let (X,τ) be a topological space then every semi - open set of (X,τ) is β *-open.

Proof: Let A be any semi - open set in X .Then A ⊆ Cl (Int(A)) ⊆ Cl (Int *(A)) ⊆ Cl (Int *(Cl(A))) .Hence A is β *-open.

Remark 3.15: The converse of the above theorem need not be true.

Example 3.16: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a},{b, c, d}}, then (X,τ) be a topological space. τ^c = {ϕ, X,{a},{b, c, d}}, then β *O(X) = {X, ϕ,{a}, {b}, {c},{d},{a, b},{a, c}, {a, d},{b, c},(b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d},{b, c, d}} and SO(X) = { X, ϕ,{a}, {b, c, d}}. But {b}, {c},{d},{a, b}, {a, c}, {a, d},{b, c},(b, d}, {c, d}, {a, b, c}, {a, b, d},{a, c, d} are β *- open but not semi - open.

Theorem 3.17: Let (X,τ) be a topological space then every semi* - open set of (X,τ) is β *- open.

Proof: Let A be any semi*- open set in X .Then A ⊆ Cl*(Int(A)) ⊆ Cl (Int (A)) ⊆ Cl(Int*(A)) ⊆ Cl (Int *(Cl(A))). Hence A is β *-open.

Remark 3.18: The converse of the above theorem need not be true.

Example 3.19: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a, b}}, then (X,τ) be a topological space. τ^c= {ϕ, X, {c, d}}, then β *O(X) = {X, ϕ,{a}, {b}, {c},{d},{a, b},{a, c}, {a, d},{b, c},(b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d},{b, c, d}} and S*O(X) = { X, ϕ, {a, b}}. Clearly the sets {a}, {b}, {c},{d},{a, c}, {a, d},{b, c},(b, d}, {c, d},{a, b, c}, {a, b, d}, {a, c, d},{b, c, d} are β *- open but not semi*- open.

Theorem 3.20: Let (X,τ) be a topological space then every pre - open set of (X,τ) is β *- open.
Proof: Let A be any pre - open set in X .Then A ⊆ Int(Cl (A)) ⊆ Int *(Cl (A)) ⊆ Cl(Int*(Cl (A))). Hence A is β *-open.

Remark 3.21: The converse of the above theorem need not be true.

Example 3.22: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a},{a, b, c}}, then (X,τ) be a topological space. τ^c = {ϕ, X,{d},{b, c, d}}, then β *O(X) = {X, ϕ,{a}, {b}, {c}, {d}, {a, b},{a, c}, {a, d},{b, c},{b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d},{b, c, d}} and PO(X) = { X, ϕ,{a},{a, b}, {a, c}, {a, d},{a, b, c}, {a, b, d}, {a, c, d}}. Clearly the sets {b}, {c}, {d}, {b, c},{b, d}, {c, d},{b, c, d} are β *- open but not pre - open.

Theorem 3.23: Let (X,τ) be a topological space then every pre*- open set of (X,τ) is β *- open.
Proof: Let A be any pre *- open set in X .Then A ⊆ Int*(Cl (A)) ⊆ Cl (Int *(Cl (A))). Hence A is β *- open.

Remark 3.24: The converse of the above theorem need not be true.

Example 3.25: Let X = {a, b, c, d} be any set and τ = {X, ϕ,{a, b},{a, b, c}}, then (X,τ) be a topological space. τ^c = {ϕ, X,{d},{c, d}}, then β *O(X) = {X, ϕ,{a}, {b}, {c},{a, b},{a, c}, {a, d},{b, c}, {b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d},{b, c, d}} and P*O(X) = { X, ϕ,{a}, {b}, {c},{a, b},{a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d},{a, c, d},{b, c, d}}. Clearly the set {c, d} is in β *- open but not pre*- open.

Theorem 3.26: Let { A_α } be a collection of β *-open sets in topological space (X,τ) then $\cup A_\alpha$ is β *-open.
Proof: Let { A_α : $\alpha \in \Delta$ } b e a collection of β *-open sets in X. Then A_α ⊆ Cl (Int *(Cl (A_α))).
$\cup A_\alpha \subseteq \cup$ Cl (Int *(Cl (A_α))) ⊆ Cl ($\cup$ Int*(Cl (A_α))) ⊆ Cl (Int* $\cup$ (Cl (A_α))) ⊆ Cl (Int* (Cl ($\cup A_\alpha$))).Therefore $\cup A_\alpha$ is β *-open.

Remark 3.27: The intersection of two β *- open sets need not be β *- open.

Example 3.28: Let X = {a, b, c} be any set and τ = {X, ϕ,{a, b}}, then (X,τ) be a topological space. τ^c = {ϕ, X,{c}}, then β *O(X) = {X, ϕ,{a}, {b}, {a, b},{a, c}, {b, c}}.The sets {a, c},{b, c} both are in β *-open but their intersection {a, c} $\cap$ {b, c} = { c} is not in β *-open.

Remark 3.29: The collection of β *O(X) does not form a topology.

Corollary 3.30: Let A and B are any two subsets of the space X, where A is β *-open and B is β - open then A $\cup$ B is β *- open.
Proof: It follows directly from the Theorems 3.8 and 3.17.

Corollary 3.31: Let A and B are any two subsets of the space X, where A is β *- open and B is g- open then A $\cup$ B is β *- open.
Proof: It follows directly from the Theorems 3.6 and 3.17.

Corollary 3.32: Let A and B are any two subsets of the space X, where A is β *-open and B is open then A $\cup$ B is β *- open.

Proof: It follows directly from the Theorems 3.5 and 3.26.
Remark 3.33: The concept of β *- open and g*- open are independent.

Example 3.34: Let X ={a, b, c, d} be any set and τ ={X, ϕ,{a}, {b}, {c}, {a, b}, {a, c},{b, c}, {a, b, c}},
β *O(X) = {X, ϕ,{a}, {b}, {c},{a, b},{a, c}, {a, d},{b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d},{b, c, d}} and g*- open = {{a}, {b}, {c},{d},{a, b},{a, c}, {b, c}, {a, b, c}}. It can be verified that {d} is in g*- open but not in β *- open.

Example 3.35: Let X ={a, b, c, d} be any set and τ ={X, ϕ,{a}, {b}, {c}, {a, b}, {a, c},{b, c}, {a, b, c}},
It can be verified that {a, d}, { b, d},{c, d},{a, b, d}, {a, c, d,},{ b, c, d} is in β * - open but not in g*- open.

Theorem 3.36: Let (X,τ) be a topological space. Then
1. Every closed set of (X,τ) is β *- closed.
2. Every β -closed set of (X,τ) is β *- closed.
3. Every g-closed set of (X,τ) is β *- closed.
4. Every semi closed set of (X,τ) is β *-closed.
5. Every semi*closed set of (X,τ) is β *-closed.
6. Every pre closed set of (X,τ) is β *closed.
7. Every pre*closed set of (X,τ) is β *-closed.
8. Let { A_α } be a collection of β *-closed sets in topological space (X,τ) then $\cap A_\alpha$ is β *- closed.
9. Let A and B are any two subsets of the space X, where A is β *-closed and B is β - closed then A $\cap$ B is β *-closed.
10. Let A and B are any two subsets of the space X, where A is β *-closed and B is g- closed then A $\cap$ B is β *-closed.
11. Let A and B are any two subsets of the space X, where A is β *-closed and B is closed then A $\cap$ B is β *-closed.

Proof: It is directly follows from the Theorems 3.5, 3.8, 3.11, 3.14, 3.17, 3.20, 3.23, 3.26, 3.30, 3.31, 3.32.

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