NEIGHBOURHOOD GRAPH OF CARTESIAN PRODUCT FOR SOME GRAPHS

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Abstract: Given a graph G, a new graph can be constructed by using Neighbourhood sets of G. This new graph is called the Neighbourhood graph of the given graph G. This paper makes a study of Neighbourhood graph of cartesian product of some graphs.

Keywords: Neighbourhood Set, Neighbourhood Number.

Introduction: Graphs discussed in this paper are simple, undirected and finite. For v ∈ V (G), the neighbourhood N(v) of v is the set of all vertices adjacent to v in G. N[v] = N(v) ∪ {v} is called closed neighbour of v. A vertex v ∈ V (G) is called support if it is adjacent to a pendant vertex (that is a vertex of degree one). Any undefined terms in this paper may be found in Harary [2].
The concept of neighbourhood number of a graph has been studied by Prof. E. Sampath kumar and Prof. P. S. Neeralagi [7]. A set S of points in a graph G is a neighbourhood set if G = $\bigcup_{v\in s} \langle N[v]\rangle$, where $\langle N[v]\rangle$ is the subgraph of G induced by v and all points adjacent to v. The neighbourhood number n_0 (G) of G is the minimum cardinality of a neighourhood set. The concept of γ - graph of a graph denoted by γ .(G) was introduced by Dr. N. Sridharan[8]. A subset D of V(G) is said to be a dominating set of G, if every vertex in V – D is adjacent to some vertex in D. The domination numbet γ (G) is the minimum cardinality of a dominating set. This parameter has been investigated by many authors including Berge, cockayne, Hedeteniemi and Walikar et al. The vertex set V(γ .(G)) is the set of all γ -sets G and for two sets,S_i, $S_j \in$ V(γ .(G)), S_i, S_j are adjacent in γ .(G) if and only if $S_j = (S_i - \{U\}) \bigcup \{v\}$, where u $\in S_i$, v $\notin S_i$, i ≠ j. In [8], Dr. N. Sridharan developed a lot of interesting results by making use of this new graph. The maximum cardinality of a maximum independent set is called the independence number of G and is denoted by β_0 (G). In [6] Dr. V. Swaminathan, A. P. Pushpalatha et al introduced the concept of β_0 .(G) (namely β_0 - graph of a graph). The vertex set V(β_0 .(G)) is the set of all β_0 - sets of G and for two sets S_i, $S_j \in$ V(β_0 .(G)), S_i, S_j are adjacent in β_0 .(G) if and only if $S_j = (S_i - \{u\}) \bigcup \{v\}$, where u $\in S_i$, v $\notin S_i$, i ≠ j.

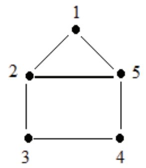
Neighbourhood Graph of a Graph:

Definition 1.1: A set S of points in a graph G is a **neighbourhood set** if G = $\bigcup_{v \in S} \langle N[v]\rangle$, where $\langle N[v]\rangle$ is the subgraph of G induced by v and all points adjacent to v. The neighbourhood number n_0 (G) of G is the minimum cardinality of a neigbourhood set and the set is denoted by n_0 - set of G. The set of all n_0 - sets of G is the vertex set of n_0 .(G) and two n_0 -sets D_1 and D_2 are adjacent in n_0 .(G) if and only if $|D_1 \cap D_2| = |D_1| - 1 = |D_2| - 1$.

Definition 1.2: Let G be a graph. Let H be a graph whose vertex set is the set of all neighbourhood sets of G and two vertices u, v $\in$ H representing neighbourhood sets S_i, S_j respectively are adjacent if and only if $S_j = (S_i - \{u\}) \bigcup \{v\}$, where u $\in S_i$, v $\notin S_i$, $i \neq j$. H is called the neighbourhood graph of a graph G and is denoted by n_0.(G).

Example 1.3:

G:



The n_0 -sets of G are {2, 4}, {3, 5}. The n_0 -graph of G is as follows:

n_0.(G):

(ie) n_0.(G) = $2K_1$ or $\overline{K}_2$.

Observation 1.4: $|V[n_0.(G)]|$ = The number of distinct n_0 - sets of G.

Neighbourhood Graph of Some Standard Graphs:

1. $n_0.(K_n) = K_1$, $\forall$ n.
2. $n_0.(K_{1,n}) = K_1$
3. $n_0.(K_{m,n}) = \begin{cases} K_1 & if\ n < m\ and\ m < n \\ 2K_1 & otherwise \end{cases}$
4. $n_0.(W_n) = K_1$
5. $n_0.(C_n) = \begin{cases} 2K_1, & if\ n\ is\ even \\ C_n, & if\ n\ is\ odd \end{cases}$
6. $n_0.(P_n) = \begin{cases} P_{m+1}, & if\ n = 2m, m \geq 1 \\ K_1, & if\ n = 2m+1 \end{cases}$
7. $n_0.(F_n) = K_1$
8. $n_0.(D_{r,s}) = K_1$
9. $n_0.(F_{m,n}) = K_1$

Neighbourhood Graph of Cartesian Product of Some Graphs:

Definition 3.1: Let $G_1 = (V_1,E_1)$ and
$G_2 = (V_2,E_2)$ be any two graphs. Then their **Cartesian product** $G_1 \square G_2$ is defined to be the graph whose vertex set is $V_1 \square V_2$ and the edge set is $\{((u_1,v_1),(u_2,v_2))$ either $u_1 = u_2$ and $v_1v_2 \in E_2$ or $v_1 = v_2$ and $u_1u_2 \in E_1\}$.

Theorem 3.2 If $G = K_n \square P_m$, then
$n_o.(K_n \square P_m) = nK_1$, where m is odd and $m \geq 3$, $n \geq 3$.

Proof: Let $G = K_n \square P_m$. Let $V(K_n) = \{u_1,u_2,...,u_n\}$ and $V(P_m) = \{v_1,v_2,...,v_m\}$.
Then $V(G) = \{(u_1,v_1),...,(u_n,v_1),(u_n,v_2)\ (u_n,v_2),...,(u_1,v_m),...,(u_n,v_m)\}$. Then the possible n_o – sets of G are:
$S_1 = A_1 \cup B_1$, where
$A_1 = \{(u_i,v_j)\ /\ i = 2,3,4,...,n$ and $i \neq 1$, j is even, $1 \leq j \leq m\}$
$B_1 = \{(u_{n-i}, v_j)/\ i = n-1$, j is odd, $1 \leq j \leq m\}$
$S_2 = A_2 \cup B_2$, where
$A_2 = \{(u_i,v_j)\ /\ i = 1,3,4,...,n$ and $i \neq 2$, j is even, $1 \leq j \leq m\}$

$B_2 = \{(u_{n-i}, v_j) / i = n-2, j \text{ is odd}, 1 \le j \le m\}$
$S_3 = A_3 \cup B_3$, where
$A_3 = \{(u_i,v_j) / i = 1,2,4,\ldots,n \text{ and } i \neq 3, j \text{ is even}, 1 \le j \le m\}$
$B_3 = \{(u_{n-i}, v_j) / i = n-3, j \text{ is odd}, 1 \le j \le m\}$
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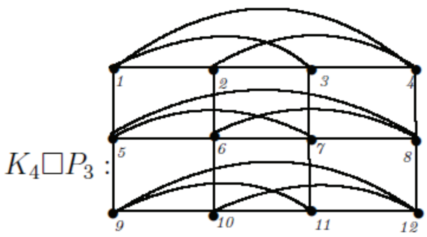
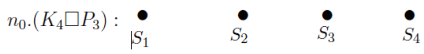
.
$S_{n-1} = A_{n-1} \cup B_{n-1}$, where
$A_{n-1} = \{(u_i,v_j) / i = 1,2,3,\ldots,n-2 \text{ and } i \neq n-1, j \text{ is even}, 1 \le j \le m\}$
$B_{n-1} = \{(u_{n-i}, v_j) / i = 1, j \text{ is odd}, 1 \le j \le m\}$
$S_n = A_n \cup B_n$, where
$A_n = \{(u_i,v_j) / i = 2,3,4,\ldots,n-1 \text{ and } i \neq n, j \text{ is even}, 1 \le j \le m\}$
$B_n = \{(u_{n-i}, v_j) / i = 0, j \text{ is odd}, 1 \le j \le m\}$
There are n – number of distinct n_o-sets of G. Hence these n_o-sets can be considered as the vertex set of n_o-graph of G. Using these points, we get $n_o.(K_n \square P_m) = nK_1$.

Example 3.3



The n_o-sets of $K_4 \square P_3$ are {5,6,7,4,12}, {5,6,8,3,11},{5,7,8,2,10},{6,7,8,1,9} namely S_1 , S_2, S_3, S_4 respectively. The n_o- graph of $K_4 \square P_3$ is as follows:



Theorem 3.4: IfG = $C_n \square P_2$, then $n_o.(C_n \square P_2) = C_{2n}$, where n is odd.
Proof: Let G = $C_n \square P_2$. Let $V(C_n) = \{u_1,u_2,\ldots,u_n\}$ and $V(P_2) = \{v_1,v_2\}$. Then $V(G)=\{(u_1,v_1),(u_2,v_1),\ldots,(u_n,v_1),(u_1,v_2), (u_2,v_2),\ldots,(u_n,v_2)\}$. Then the possible n_o-sets of G are,
$S_1 = \{A_1 \cup B_1 \cup C_1\}$, where

$A_1 = \{(u_i,v_i) / i = 1\}, B_1 = \{(\bigcup_{j=2}^{n-1} u_j, v_1) / j \text{ is even}\}, C_1 = \{(\bigcup_{r=1}^{n} u_r, v_2) / r \text{ is odd}\}.$

$S_1 = \{A_1 \cup B_1 \cup C_1\}$, where

$A_1 = \{(u_i,v_i) / i = 1\}, B_1 = \{(\bigcup_{j=2}^{n-1} u_j, v_1) / j \text{ is even}\}, C_1 = \{(\bigcup_{r=1}^{n} u_r, v_2) / r \text{ is odd}\}.$

$S_2 = \{A_2 \cup B_2 \cup C_2 \cup D_2\}$, where

$A_2 = \{(u_i,v_1) / i = 1\}, B_2 = \{(\bigcup_{j=2}^{n-1} u_j, v_1) / j \text{ is even}\}, C_1 = \{(\bigcup_{r=3}^{n} u_r, v_2) / r \text{ is odd}\},$

$D_2 = \{(u_s,v_2)/s=2\}$.
$S_3 = \{A_3 \cup B_3 \cup C_3 \cup D_3\}$, where

$A_3 = \{(u_i,v_1) / i = 1,3\}, B_3 = \{(\bigcup_{j=4}^{n-1} u_j, v_1) / j \text{ is even}\}, C_3 = \{(\bigcup_{r=3}^{n} u_r, v_2) / r \text{ is odd}\},$

$D_3 = \{(u_s,v_2)/s=2\}$.
$S_4 = \{A_4 \cup B_4 \cup C_4 \cup D_4\}$, where

$A_4 = \{(u_i,v_1)/ i = 1,3\}$, $B_4 = \{(\bigcup_{j=4}^{n-1} u_j , v_1)/ j \text{ is even}\}$, $C_4 = \{(\bigcup_{r=5}^{n} u_r , v_2)/ r \text{ is odd}\}$,

$D_4 = \{(u_s,v_2)/s=2,4\}$.

⋮

$S_{n-4} = \{A_{n-4} \cup B_{n-4} \cup C_{n-4} \cup D_{n-4}\}$, where

$A_{n-4} = \{(\bigcup_{i=1}^{n-4} u_i , v_1)/ i \text{ is odd}\}$, $B_{n-4} = \{(\bigcup_{j=n-3}^{n-1} u_j , v_1)/ j \text{ is even}\}$, $C_{n-4} = \{(\bigcup_{r=n-4}^{n} u_r , v_2)/ r \text{ is odd}\}$, $D_{n-4} = \{(\bigcup_{s=2}^{n-5} u_s , v_2)/ s \text{ is even}\}$.

⋮

$S_{n-2} = \{A_{n-2} \cup B_{n-2} \cup C_{n-2} \cup D_{n-2}\}$, where

$A_{n-2} = \{(\bigcup_{i=1}^{n-2} u_i , v_1)/ i \text{ is odd}\}$, $B_{n-2} = \{(u_j,v_1)/ j=n-1\}$, $C_{n-2} = \{(\bigcup_{r=n-2}^{n} u_r , v_2)/ r \text{ is odd}\}$, $D_{n-2} = \{(\bigcup_{s=2}^{n-3} u_s , v_2)/ s \text{ is even}\}$.

$S_{n-1} = \{A_{n-1} \cup B_{n-1} \cup C_{n-1} \cup D_{n-1}\}$, where

$A_{n-1} = \{(\bigcup_{i=1}^{n-2} u_i , v_1)/ i \text{ is odd}\}$, $B_{n-1} = \{(u_j,v_1)/ j=n-1\}$, $C_{n-1} = \{(u_r,v_2)/ r = n\}$, $D_{n-1} = \{(\bigcup_{s=2}^{n-1} u_s , v_2)/ s \text{ is even}\}$.

$S_{n} = \{A_n \cup C_n \cup D_n\}$, where

$A_n = \{(\bigcup_{i=1}^{n} u_i , v_1)/ i \text{ is odd}\}$, $C_n = \{(u_r,v_2)/ r = n\}$, $D_n = \{(\bigcup_{s=2}^{n-1} u_s , v_2)/ s \text{ is even}\}$.

$S_{n+1} = \{A_{n+1} \cup C_{n+1} \cup D_{n+1}\}$, where

$A_{n+1} = \{(\bigcup_{i=1}^{n} u_i , v_1)/ i \text{ is odd}\}$, $C_{n+1} = \{(\bigcup_{r=2}^{n-2} u_r , v_2)/ r \text{ is even}\}$, $D_{n+1} = \{(u_s,v_2)/ s = 1\}$.

$S_{n+2} = \{A_{n+2} \cup B_{n+2} \cup C_{n+2} \cup D_{n+2}\}$, where

$A_{n+2} = \{(u_i,v_1)/ i= 2\}$, $B_{n+2} = \{(\bigcup_{j=3}^{n} u_j , v_1)/ j \text{ is odd}\}$, $C_{n+2} = \{(\bigcup_{r=2}^{n-1} u_r , v_2)/ r \text{ is even}\}$, $D_{n+2} = \{(u_s,v_2)/ s = 1\}$.

$S_{n+3} = \{A_{n+3} \cup B_{n+3} \cup C_{n+3} \cup D_{n+3}\}$, where

$A_{n+3} = \{(u_i,v_1)/ i= 2\}$, $B_{n+3} = \{(\bigcup_{j=3}^{n} u_j , v_1)/ j \text{ is odd}\}$, $C_{n+3} = \{(\bigcup_{r=n-5}^{n-1} u_r , v_2)/ r \text{ is even}\}$, $D_{n+3} = \{(u_s,v_2)/ s = 1,3\}$.

⋮

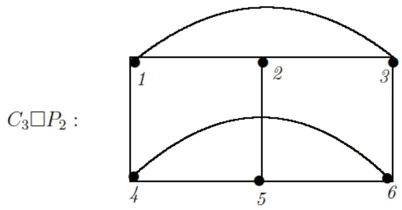
$S_{2n-5} = \{A_{2n-5} \cup B_{2n-5} \cup C_{2n-5} \cup D_{2n-5}\}$, where $A_{2n-5} = \{(\bigcup_{i=2}^{n-5} u_i , v_1)/ i \text{ is odd}\}$, $B_{2n-5} = \{(\bigcup_{j=n-4}^{n} u_j , v_1)/ j \text{ is even}\}$, $C_{2n-5} = \{(\bigcup_{r=n-5}^{n-1} u_r , v_2)/ r \text{ is odd}\}$, $D_{2n-5} = \{(\bigcup_{s=1}^{n-6} u_s , v_2)/ s \text{ is even}\}$.

$S_{2n-2} = \{A_{2n-2} \cup B_{2n-2} \cup C_{2n-2} \cup D_{2n-2}\}$, where $A_{2n-2} = \{(\bigcup_{i=2}^{n-3} u_i , v_1)/ i \text{ is even}\}$, $B_{2n-2} = \{(\bigcup_{j=n-2}^{n} u_j , v_1)/ j \text{ is odd}\}$, $C_{2n-2} = \{(u_r,v_2)/ r = n - 1\}$, $D_{2n-2} = \{(\bigcup_{s=1}^{n-2} u_s , v_2)/ s \text{ is odd}\}$.

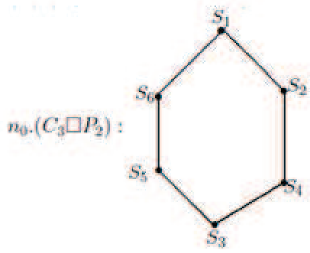
$S_{2n} = \{A_{2n} \cup B_{2n} \cup D_{2n}\}$, where $A_{2n} = \{(\bigcup_{i=2}^{n-1} u_i , v_1)/ i \text{ is even}\}$, $B_{2n} = \{(\bigcup_{j=n-2}^{n} u_j , v_1)/ j = n\}$, $D_{2n} = \{(\bigcup_{s=1}^{n} u_s , v_2)/ s$

is odd}.
Then clearly we get 2n number of distinct n_o-sets namely $S1,S2,\ldots,S_{2n}$.
To construct $n_o.(C_n \square P_2)$, these 2n n_o-sets are considered as the vertices of $n_o(G)$. Then any pair of different n_o-sets say S_i,S_j, $1 \leq i,j \leq 2n$, $i \neq j$ are differe exactly in one place, hence these n_o-sets are obviously adjacent in $n_o.(G)$. Then the set $S1,S2,\ldots,S_{2n}$ form a cycle with the order 2n. Hence $n_o.(C_n \square P_2) = C_{2n}$.

Example 3.5:

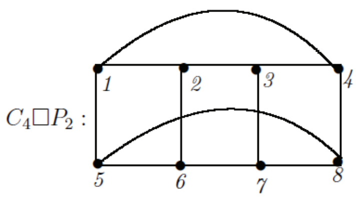


The n_o-sets of $C_3 \square P_2$ are $S_1=\{1,2,4,6\}$,
$S_2 = \{1,2,4,5\}$, $S_3 = \{1,3,5,4\}$, $S_4 = \{1,3,5,6\}$, $S_5 = \{2,3,4,5\}$, $S_6 = \{2,3,4,6\}$.
Then the n_o – graph of $C_n \square P_2$ is as follows:



Theorem 3.6: If $G = C_n \square P_2$, then $n_o.(C_n \square P_2) = 2K_1$, where n is even.
Proof: Assume that $G = C_n \square P_2$. Let $V(C_n) = \{u_1,u_2,\ldots,u_n\}$ and $V(P_2) = \{v_1,v_2\}$. So $V(G)=\{(u_1,v_1),(u_2,v_1),\ldots,(u_n,v_1),(u_1,v_2), (u_2,v_2),\ldots,(u_n,v_2)\}$.
Then G has exactly two disjoint n_o-sets of cardinality ' n' namely,
$S_1 = \{(u_i,v_1) \cup (u_j,v_2)$ / i is odd, $1 \leq n$, j is even, $1 \leq j \leq n\}$
$S_2 = \{(u_j,v_1) \cup (u_i,v_2)$ / i is odd, $1 \leq n$, j is even, $1 \leq j \leq n\}$.
Hence $n_o.(C_n \square P_2) = 2K_1$.

Example 3.7:



The n_o-sets of $C_4 \square P_2$ are $S_1 = \{1,3,6,8\}$, $S_2 = \{2,4,5,7\}$. Then the n_o-graph of $C_4 \square P_2$ is as follows:

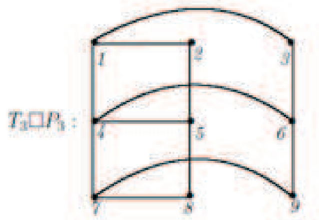
Theorem 3.8: If T_n is a tree with maximum number of independent vertices and P_m is a path, where $m \geq 3$, then

$$n_o.(T_n \square P_m) = \begin{cases} 2K_1, if \ m \ is \ even \\ K_1, if \ m \ is \ odd \end{cases}$$

Proof: Let T_n a tree with maximum number of independent vertices and P_m be a path.

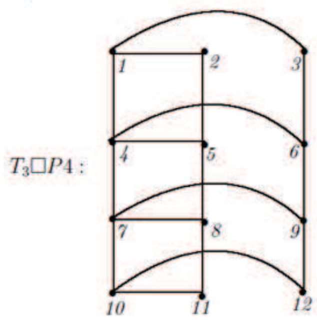
Case:1 m is even.
If m is even, then $T_n\Box P_m$ has exactly two disjoint n_o-sets. (i.e) $S_1 = A \cup B$ and $S_2 = C \cup D$, where
$A = \{(u_1,v_i)/ 1 \leq i \leq m, i \text{ is odd}\}$
$B=\{(u_i,v_j) /i = 2,3,\ldots,n, 1 \leq j\leq m, j \text{ is even}\}$
$C = \{(u_1,v_i)/ 1 \leq i \leq m, i \text{ is even}\}$,
$D = \{(u_i,v_j) /i = 2,3,\ldots,n, 1 \leq j\leq m, j \text{ is odd}\}$. Hence $n_o.(T_n\Box P_m) = 2K_1$.
Case:2 m is odd.
If m is odd, then $T_n\Box P_m$ has a unique set. (i.e) $S= A \cup B$, where
$A = \{(u_1,v_i)/ i = 1,2,\ldots m, i \text{ is odd}\}$
$B = \{(u_i,v_j) /i = 2,3,\ldots,n, 1 \leq j\leq m, j \text{ is even}\}$. Hence $n_o.(T_n\Box P_m) = K_1$.

Example 3.9:



The n_o-sets of $T_3\Box P_3$ is {1,5,6,7}. Hence $n_o.(T_3\Box P_3) = K_1$.

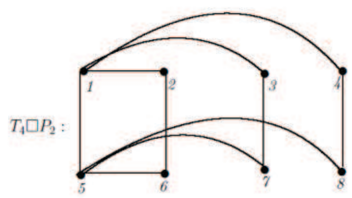
Example 3.10:



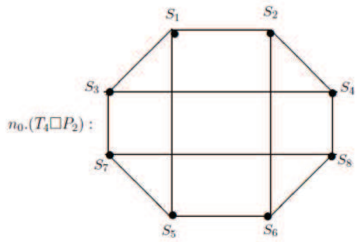
The n_o-sets of $T_3\Box P_4$ are {1,5,6,7,11,12} and {4,10,8,9,2,3}. $n_o.(T_3\Box P_4) = 2K_1$.

Result 3.11: If $T_{n+1} = K_{1,n}$,then $n_o.(T_{n+1}\Box P_2)$ is $(n - 1)$ – regular graph with the order 2^n.

Example 3.12:



The n_o- sets of $T_4\Box P_2$ are namely {1,5,2,3,4},{1,5,2,3,8},{1,5,2,4,7},{1,5,3,4,6},{1,5,3,6,8},{1,5,4,6,7} namely S_1, $S_2,\ldots,S_8$ respectively. The n_o-graph of $T_4\Box P_2$ is as follows:

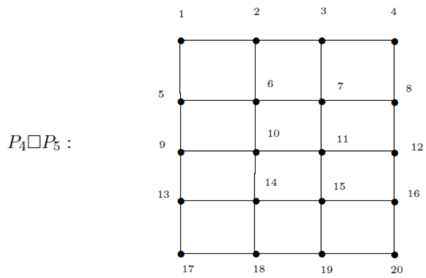


Theorem 3.13 If If $mn\equiv 0 \pmod 2$, then $n_o.(P_m\square\, P_n) = 2K_1$. Otherwise $n_o.(P_m\square\, P_n) = K_1$.

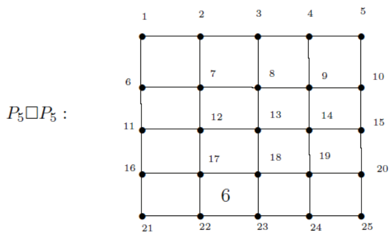
Proof: If $mn\equiv 0 \pmod 2$, then $P_m\square\, P_n$ has exactly two disjoint minimum n_o-sets of cardinality $\frac{mn}{2}$. Hence $n_o.(P_m\square\, P_n) = 2K_1$.

If $mn\equiv 1 \pmod 2$, then has $P_m\square\, P_n$ a unique n_o-setsof cardinality $\left\lfloor \frac{mn}{2} \right\rfloor$.Hence $n_o.(P_m\square\, P_n) = K_1$.

Example 3.14 :
Case: 1 $mn\equiv 0 \pmod 2$.



The n_o-sets of $P_4\square P_5$ are {2,4,6,8,10,12,14,16,18,20}, (1,3,5,7,9,11,13,15,17,19}. Therefore $n_o.(P_4\square\, P_5) = 2K_1$.
Case:2 $mn\equiv 1 \pmod 2$.



The n_o-sets of $P_5\square P_5$ is {2,4,6,8,10,12,14,16,18,20,22,24}. Hence $n_o.(P_5\square\, P_5) = K_1$.

Conclusion: In this paper, we have made a study of new concept called neighbourhood graph of a graph. It is further continued in our subsequent investigations in this direction.

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