CONNECTEDNESS AND COMPACTNESS VIA v-OPEN SETS IN TOPOLOGICAL SPACES

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Abstract: The purpose of this paper is to introduce the concepts of v-connected spaces and v-compact spaces and to define v-separated sets using v-closure operator. We investigate some of their basic properties. We also discuss their relationship with already existing concepts.

Keywords: v-Connected, v-Separated, v-Compact.

Mathematics Subject classification 2010: 54A05, 54D05, 54D30.

1. Introduction: Generalized closed sets in a topological space is introduced by Levine[3] in 1970. In 1963 Levine [2] introduced semi-open sets in topological spaces. After Levine's work, many mathematicians turned their attention to generalizing various concepts in topology by considering semi-open sets instead of open sets. Dunham [1] introduced the concept of generalized closure using Levine's generalized closed sets and defined a new topology τ* and studied some of their properties. The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness and connectedness. The productivity and fruitfulness of these notions of compactness and connectedness motivated mathematicians to generalize these notions. In the course of these attempts many stronger and weaker forms of compactness and connectedness have been introduced and investigated.

In this paper, we introduce the concepts of v-connected spaces and v-compact spaces and define v-separated sets using v-closure operator. We investigate some of their basic properties. We also discuss their relationship with already existing concepts.

2. Preliminaries: Throughout this paper (X, τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. (X, τ) will be replaced by X if there is no changes of confusion. For a subset A of a topological space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A respectively. We recall the following definitions and results.

Definition 2.1: A subset A of a topological space (X, τ) is called generalized closed [3] (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open X. A subset A of a topological space (X, τ) is called generalized open (briefly g-open) if $X \backslash A$ is g-closed.

Definition 2.2: A subset A of a topological space (X, τ). The generalized closure of A[1] is defined as the intersection of all g-closed sets containing A and is denoted by $cl^*(A)$. The generalized interior of A is defined as the union all g-open sets contained in A and it is denoted by $int^*(A)$.

Definition 2.3: A subset A of a topological space (X,τ) is said to be a v-open set[5] if $A \subseteq int^*(cl(A)) \cup cl^*(int(A))$. The collection of all v-open sets in (X,τ) is denoted by $v - O(X,\tau)$. A subset A of a topological space (X,τ) is called a v-closed set if $X \backslash A$ is v-open. The collection of all v-closed sets in (X,τ) is denoted by $v - C(X,\tau)$.

Theorem 2.4 [5]: Every open set is v-open.

Definition 2.5: Let A be a subset of a topological space (X,τ). Then the union of all v-open sets contained in A is called the v-interior of A[6] and it is denoted by $vint(A)$. That is, $vint(A) = \{V : V \subseteq A \text{ and } V \in v - O(X)\}$.

Definition 2.6: Let A be a subset of a topological space (X,τ). Then the intersection of all v-closed sets in X containing A is called the v-closure of A[6] and it is denoted by $vcl(A)$. That is, $vcl(A) = \cap \{F : A \subseteq F \text{ and } F \in v - C(X)\}$.

Definition 2.7: A function $f : X \longrightarrow Y$ is called a v-continuous[7] if the inverse image of each open set in Y is v-open in X.

Definition 2.8: A function $f : X \longrightarrow Y$ is called v-irresolute[7] if the inverse image of every v-open set in Y is v-open in X.

Definition 2.9: A function $f : X \longrightarrow Y$ is said to be v-closed[7] if image of each closed set in X is v-closed in Y.

Definition 2.10: A function $f : X \longrightarrow Y$ is said to be v-open[7] if image of each open set in X is v-open in Y.

3. v-Connected Spaces: In this section we introduce v-connected spaces. We give characterizations for v-connected spaces and also investigate their basic properties.

Definition 3.1: A topological space (X,τ) is said to be v-connected if X cannot be expressed as the union of two disjoint non-empty v-open sets.

Definition 3.2: A subset S of a topological space (X,τ) is said to be a v-connected set in X if S cannot be expressed as the union of two disjoint non-empty v-open sets in X.

Theorem 3.3: If a space X is v-connected then it is connected.
Proof: Let X be v-connected. Suppose X is not connected. Then there exist disjoint non-empty open sets A and B such that $X = A \cup B$. By Theorem 2.2, A and B are v-open sets. This is a contradiction to X is v-connected.
However, the reverse implications of Theorem 3.3 is not true as shown by the following example.

Example 3.4: Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, \{b\}, \{a,b\}, X\}$. Here $v - O(X,\tau) = \{\phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}$. Clearly X is connected. But not v-connected, since X is the union of two disjoint non-empty v-open sets $\{a\}$ and $\{b,c\}$.

Theorem 3.5: For a topological space X the following are equivalent.
(i) X is v-connected.
(ii) The only subsets of X which are both v-open and v-closed are the empty set and X.
(iii) Each v-continuous of X into a discrete space Y with at least two points is a constant map.
Proof: (i) ⇒ (ii): Let X be a v-connected space. Suppose A is a subset of X which is both v-open and v-closed. Then A and $X \backslash A$ are disjoint v-open sets and $X = A \cup (X \backslash A)$. Since X is v-connected, either $A = \phi$ or $X \backslash A = \phi$. That is either $A = \phi$ or $A = X$.

(ii) ⇒ (i): Suppose X is not v-connected. Then there are disjoint non-empty proper v-open sets A and B such that $X = A \cup B$. Since $A = X\backslash B$ and A is v-closed, by our assumption $A = \phi$ or $A = X$ which is a contradiction to A is a non-empty proper v-open set. Therefore X is v-connected.
(ii) ⇒ (iii): Let $f : X \to Y$ be a v-continuous function, where Y is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is both v-open and v-closed for each $y \in Y$. By (ii), $f^{-1}(\{y\}) = \phi$ or X. If $f^{-1}(\{y\}) = \phi$ for all $y \in Y$, then f will not be a function. Also there cannot be exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence there exist only one $y_1 \in Y$ such that $f^{-1}(\{y_1\}) = X$ and hence $f(X) = \{y_1\}$. This shows that f is a constant map.
(iii) ⇒ (ii): Let S be both a v-open and a v-closed set in X. Suppose $S \neq \phi$. Let Y be a discrete space with at least two points. Fix y_0 and y_1 in Y and $y_0 \neq y_1$. Define $f: X \to Y$ by $f(x) = y_0$ for $x \in S$ and y_1 for $x \notin S$. Let V be an open set in Y. If V contains y_0 alone, then $f^{-1}(V) = S$. If V contains y_1 alone, then $f^{-1}(V) = X\backslash S$. If V contains both y_0 and y_1, then $f^{-1}(V) = X$. Otherwise $f^{-1}(V) = \phi$. In all the cases $f^{-1}(V)$ is v-open in X. Therefore f is v-continuous function. By (iii) f is constant. Therefore $f(x) = y_0$ or $f(x) = y_1$ for all x in X. If $f(x) = y_0$ for all x in X, then $S = X$. If $f(x) = y_1$ for all x in X, then $S = \phi$. These complete the theorem.

Definition 3.6: Two non-empty subsets A and B of a space X are called v-separated if $vcl(A) \cap B = A \cap vcl(B) = \phi$.

Theorem 3.7: Any two disjoint non-empty sets are v-closed sets if and only if they are v-separated.
Proof: Suppose A and B are disjoint non-empty v-closed sets. Then $vcl(A) \cap B = A \cap vcl(B) = A \cap B = \phi$. This shows that A and B are v-separated.

Theorem 3.8: (i) If A and B are v-separated and $C \subseteq A$, $D \subseteq B$, then C and D are also v-separated.
(ii)If A and B are both v-open and if $H = A \cap (X\backslash B)$ and $G = B \cap (X\backslash A)$, then H and G are v-separated.
Proof: (i) Let A and B be v-separated. Then $vcl(A) \cap B = \phi = A \cap vcl(B)$. Since $C \subseteq A$ and $D \subseteq B$, then $vcl(C) \subseteq vcl(A)$ and $vcl(D) \subseteq vcl(B)$. This implies that, $vcl(C) \cap D \subseteq vcl(A) \cap B = \phi$ and hence $vcl(C) \cap D = \phi$. Similarly $vcl(D) \cap C \subseteq vcl(B) \cap A = \phi$ and hence $vcl(D) \cap C = \phi$. Therefore C and D are v-separated.
(ii) Let A and B both v-open subsets in X. Then $X\backslash A$ and $X\backslash B$ are v-closed. Since $H \subseteq X\backslash B$, then $vcl(H) \subseteq vcl(X\backslash B) = X\backslash B$ and so $vcl(H) \cap B = \phi$. Since $G \subseteq B$, then $vcl(H) \cap G \subseteq vcl(H) \cap B = \phi$. Thus, $vcl(H) \cap G = \phi$. Similarly, $vcl(G) \cap H = \phi$. Hence H and G are v-separated.

Theorem 3.9: The sets A and B of a space X are v-separated if and only if there exist U and V in $v-O(X)$ such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$ and $B \cap U = \phi$.
Proof: Necessity: Let A and B be v-separated. Then $A \cap vcl(B) = \phi = vcl(A) \cap B$. Take $V = X\backslash vcl(A)$ and $U = X\backslash vcl(B)$. Then U and V are v-open sets such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$ and $B \cap U = \phi$.
Sufficiency: Let U and V be v-open sets such that $A \subseteq U$, $B \subseteq V$ and $A \cap V = \phi$, $B \cap U = \phi$. Then $A \subseteq X\backslash V$ and $B \subseteq X\backslash U$ and $X\backslash V$ and $X\backslash U$ are v-closed. This implies, $vcl(A) \subseteq vcl(X\backslash V) = X\backslash V \subseteq X\backslash B$ and $vcl(B) \subseteq vcl(X\backslash U) = X\backslash U \subseteq X\backslash A$. That is, $vcl(A) \subseteq X\backslash B$ and $vcl(B) \subseteq X\backslash A$. Therefore $A \cap vcl(B) = \phi = vcl(A) \cap B$ and hence A and B are v-separated.

Remark 3.10: Each two v-separated sets are always disjoint.
Proof: Let A and B be v-separated. Then $A \cap vcl(B) = \phi = vcl(A) \cap B$. Now, $A \cap B \subseteq A \cap vcl(B) = \phi$. Therefore $A \cap B = \phi$ and hence A and B are disjoint.

Theorem 3.11: A topological space X is v-connected if and only if X is not the union of any two v-separated sets.
Proof: Necessary: X is a v-connected space.Let Suppose $X = A \cup B$, where A and B are v-separated sets. By Definition 3.6, $vcl(A) \cap B = A \cap vcl(B) = \phi$. Since $A \subseteq vcl(A)$, we have $A \cap B \subseteq vcl(A) \cap B = \phi$. Therefore $vcl(A) \subseteq X\backslash B = A$ and $vcl(B) \subseteq X\backslash A = B$. Hence $A = vcl(A)$ and $B = vcl(B)$. Therefore A and B are v-closed sets and hence $A = X\backslash B$ and $B = X\backslash A$ are disjoint v-open sets. That is X is not v-

connected, which is a contradiction to X is a v-connected space. Hence X is not the union of any two v-separated sets.
Sufficiency: Assume that X is not the union of any two v-separated sets. Suppose X is not v-connected. Then $X = A \cup B$, where A and B are non-empty disjoint v-open sets in X. Since $A \subseteq X \backslash B$ and $B \subseteq X \backslash A$, $vcl(A) \cap B \subseteq (X \backslash B) \cap B = \phi$ and $A \cap vcl(B) \subseteq A \cap (X \backslash A) = \phi$. That is A and B are v-separated sets. This is a contradiction to (ii). Therefore X is v-connected.

Theorem 3.12: If $A \subseteq H \cup K$, where A is a v-connected set and H, K are v-separated sets, then either $A \subseteq H$ or $A \subseteq K$.
Proof: Suppose $A \nsubseteq H$ and $A \nsubseteq K$. Let $A_1 = H \cap A$ and $A_2 = K \cap A$. Then A_1 and A_2 are non-empty sets and $A_1 \cup A_2 = (H \cap A) \cup (K \cap A) = (H \cup K) \cap A = A$, because $A \subseteq H \cup K$. Since $A_1 \subseteq H$, $A_2 \subseteq K$ and H, K are v-separated sets, $vcl(A_1) \cap A_2 \subseteq vcl(H) \cap K = \phi$ and $A_1 \cap vcl(A_2) \subseteq H \cap vcl(K) = \phi$. Therefore A_1, A_2 are v-separated sets such that $A = A_1 \cup A_2$. Hence by Theorem 3.13, A is not v-connected. This is a contradiction to A is v-connected. So we get either $A \subseteq H$ or $A \subseteq K$.

Theorem 3.13: If A is v-connected set, then $vcl(A)$ is also a v-connected set.
Proof: Let A be a v-connected set. Suppose $vcl(A)$ is not v-connected. Then by Theorem 3.11, there exist v-separated sets H and K such that $vcl(A) = H \cup K$. Since A is a v-connected set and $A \subseteq vcl(A) = H \cup K$, by Theorem 3.12, either $A \subseteq H$ or $A \subseteq K$. If $A \subseteq H$, then $vcl(A) \subseteq vcl(H)$. Since H and K are v-separated sets we have $H \neq \phi$, $K \neq \phi$, and $vcl(A) \cap K \subseteq vcl(H) \cap K = \phi$ and hence $K \subseteq X \backslash vcl(A)$. Also $K \subseteq H \cup K = vcl(A)$. Therefore $K \subseteq (X \backslash vcl(A)) \cap vcl(A) = \phi$, which is a contradiction to $K \neq \phi$. Similarly if $A \subseteq K$, we get a contradiction to $H \neq \phi$. Therefore $vcl(A)$ is a v-connected set.

Theorem 3.14: If A and B are v-connected subspace of a space X such that $A \cap B \neq \phi$, then $A \cup B$ is a v-connected subspace of X.
Proof: Suppose that $A \cup B$ is not v-connected. Then there exist two v-separated sets H, K such that $A \cup B = H \cup K$, by Theorem 3.11. Since H and K are v-separated, H and K are non-empty sets and $H \cap K \subseteq vcl(H) \cap K = \phi$. Since $A \subseteq A \cup B = H \cup K$, $B \subseteq A \cup B = H \cup K$ and A, B are v-connected, by Theorem 3.12, $A \subseteq H$ or $A \subseteq K$ and $B \subseteq H$ or $B \subseteq K$.

(1) If $A \subseteq H$ and $B \subseteq H$, then $A \cup B \subseteq H$ and so $A \cup B = H$. Since H and K are disjoint, we have $K = \phi$, which is a contradiction to $K \neq \phi$. Similarly, if $A \subseteq K$ and $B \subseteq K$, we get a contradiction.

(2) If $A \subseteq H$ and $B \subseteq K$, then $A \cap B \subseteq H \cap K = \phi$. Therefore $A \cap B = \phi$, which is a contradiction to $A \cap B \neq \phi$. By the same way we can get a contradiction if $A \subseteq K$ and $B \subseteq H$.

Therefore $A \cup B$ is v-connected subspace of a space X.

Theorem 3.15: (i) Let $f: X \to Y$ be v-continuous surjection and X be v-connected. Then Y is connected.
(ii) Let $f: X \to Y$ be v-irresolute surjection and X be v-connected. Then Y is v-connected.
Proof: (i) Suppose Y is not connected. Then $Y = A \cup B$, where A and B are disjoint non-empty open sets in Y. Since f is v-continuous surjection, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty v-open sets in X. This is a contradiction to X is v-connected. Hence Y is connected.
(ii) Suppose Y is not v-connected. Then $Y = A \cup B$, where A and B are disjoint non-empty v-open sets in Y. Since f is v-irresolute surjection, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty v-open sets in X. This is a contradiction to X is v-connected. Hence Y is v-connected.

Theorem 3.16: Let $f: X \to Y$ be v-open, v-closed and injection and Y be v-connected. Then X is connected.
Proof: Let A be a clopen subset of X. Since f is v-open and v-closed, $f(A)$ is both v-open and v-closed. Since Y is v-connected by Theorem 3.6(ii), $f(A) = \phi$ or $f(A) = Y$. Hence $A = \phi$ or $A = X$, since f is injection. By Theorem 3.5, X is connected.

4. v-Compact Spaces: In this section we introduce the concept v-compact spaces using v-open sets and study some of their basic properties.
Definition 4.1: A collection $\mathcal{B}$ of v-open sets in X is called a v-open cover of a subset B of X if $B \subseteq \bigcup\{U\alpha : U\alpha \in \mathcal{B}\}$.

Definition 4.2: A topological space X is said to be v-compact if every v-open cover of X has a finite subcover.

Definition 4.3: A subset A of a topological space X is said to be v-compact relative to X if every v-open cover of X has a finite subcover.

Theorem 4.4: Every v-compact space is compact.
Proof: Let X be v-compact. Suppose X is not compact. Then there exists a open cover $\mathcal{B}$ of X has no finite subcover. Since every open set is v-open, then we have v-open cover $\mathcal{B}$ of X, which has no finite subcover. This is a contradiction to X is v-compact. Hence X is compact.

Theorem 4.5: A v-closed subset of a v-compact space X is v-compact relative to X.
Proof: Let A be a v-closed subset of a v-compact space X. Then $X \backslash A$ is v-open. Let $\mathcal{B}$ be a v-open cover of X. Then $\mathcal{B} \cup \{X \backslash A\}$ is a v-open cover of X. Since X is v-compact, it has a finite subcover say $\{P_1, P_2, \ldots, P_n, X \backslash A\}$. Then $\{P_1, P_2, \ldots, P_n\}$ is a finite v-open cover. Thus A is v-compact relative to X.

Theorem 4.6: Let $f : X \to Y$ be v-continuous surjection and X be v-compact. Then Y is compact.
Proof: Let $f: X \to Y$ be a v-continuous surjection and X be v-compact. Let $\{V_\alpha\}$ be an open cover for Y. Since f is v-continuous, $\{f^{-1}(V_\alpha)\}$ is a v-open cover of X. Since X is v-compact, $\{f^{-1}(V_\alpha)\}$ contains a finite subcover, namely $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \ldots, f^{-1}(V_{\alpha_n})\}$. Then $\{V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$is a finite subcover for Y, since f is surjection. Thus Y is compact.

Theorem 4.7: Let $f: X \to Y$ be a v-open function and Y be v-compact. Then X is compact.
Proof: Let $f: X \to Y$ be a v-open function and Y be v-compact. Let $\{V_\alpha\}$ be an open cover for X. Since f is v- open, $\{f(V_\alpha)\}$ is a v-open cover of Y. Since Y is v-compact, $\{f(V_\alpha)\}$ contains a finite subv-cover, namely $\{f(V_{\alpha 1}), f(V_{\alpha 2}), \ldots, f(V_{\alpha n})\}$. Then $\{V_{\alpha 1}, V_{\alpha 2}, \ldots, V_{\alpha n}\}$ is a finite subcover for X. Thus X is compact.

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