DOUBLY GEODETIC DOMINATION NUMBER OF A GRAPH

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Abstract: For a connected graph $G(V(G), E(G))$, a set $S \subseteq V(G)$ is called a geodetic dominating set of G if S is both a geodetic and a dominating set of G. The geodetic domination number $\gamma_g(G)$ of G is the minimum cardinality of a geodetic dominating set in G. In this paper, we introduce a new variation called doubly geodetic domination number of a graph. A set $S \subseteq V(G)$ is called a doubly geodetic set of G if each vertex not in S lies on at least two distinct geodesics of vertices in S. The doubly geodetic set of G is called a doubly geodetic dominating set if it is also a dominating set of G. The minimum cardinality of doubly geodetic dominating set is called the doubly geodetic domination number and is denoted by $\gamma_{\ddot{d}g}(G)$. In this paper, we have shown that doubly geodetic domination problem is NP-complete. Also, certain characterization and realization results are discussed.

Keywords: Doubly Geodetic Domination Number, Doubly Geodetic Number, Doubly Geodetic Set, Domination Number.

Introduction: Let $G = (V(G), E(G))$ be a connected graph without loops and multiple edges and let the order of G be n. The distance $d(u, v)$ is the length of the shortest $u - v$ path in G. An $u - v$ geodesic is an $u - v$ path of length $d(u, v)$ in G. A vertex x is said to lie on an $u - v$ geodesic P if x is an internal vertex of P. The eccentricity $e(u)$ of a vertex u is defined by $e(u) = max \{ d(u, v\} : v \in V\}$.

The minimum and the maximum eccentricity among vertices of G is its radius r and diameter d, respectively. For graph theoretic notation and terminology, we follow [1],[12].

In [1] a graph theoretical parameter called the geodetic number of a graph is introduced and it was further studied in [2],[3],[4],[8]. In [3] the geodetic number of a graph is as follows, let $I(u, v)$ be the set of all vertices lying on some $u - v$ geodesic of G. If for some non empty subset S of $V(G)$, $I(S) = \cup_{u,v \in S} I(u, v)$then S is called a geodetic set of G. A geodetic set of minimum cardinality is called minimum geodetic set of G. The cardinality of the minimum geodetic set of G is the geodetic number $g(G)$ of G. The problem of finding geodetic number of a graph is shown to be an NP-hard problem in [8]. In [5] the geodetic number is also referred as geodomination number. Chartrand, Harary Swart and Zhang were the first to study the geodetic concepts in relation to domination.

For a graph $G = (V(G), E(G))$, a set $D \subseteq V(G)$ is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of the dominating set of G [10]. A set S of vertices of a graph G is said to be geodetic dominating set if S is both a geodetic set as well as a dominating set. The minimum cardinality of a geodetic dominating set of G is its geodetic domination number, and is denoted by $\gamma_g(G)$. The geodetic domination number of a graph is defined and studied in [7].

Doubly Geodetic Domination Number: In the section we define the doubly geodetic number of a graph . A set $S \subseteq V(G)$ is called a doubly geodetic set of G if each vertex in $V - S$ lies on at least two distinct geodesics of vertices in S. The doubly geodetic number $\ddot{d}g(G)$is the minimum cardinality of a doubly geodetic set[6].

The doubly geodetic set S of G is called a doubly geodetic dominating set if S is also a dominating set of G. The minimum cardinality of a doubly geodetic dominating set is called the doubly geodetic domination number and is denoted by $\gamma_{\ddot{d}g}(G)$.

Main Results:

Theorem 3.1: For a connected graph G of order $n \geq 2$,
$2 \leq max\{\ddot{d}g(G), \gamma(G)\} \leq \gamma_{\ddot{d}g}(G) \leq n$

Corollary 3.2: The doubly geodetic domination number for a cycle C_n, where $n \geq 4$ is
$\gamma_{\ddot{d}g}(G) =$ $max\{4, \gamma(C_n)\}$ if n is odd
$max\{5, \gamma(C_n)\}$ if n is even

Theorem 3.3: The doubly geodectic domination number for a path P_n where $n \geq 3$ is $\gamma_{\ddot{d}g}(G) = max\{3, \left\lceil\frac{n+2}{3}\right\rceil\}$

Theorem 3.4: If $G = K_n$ or $(K_n - \{e\})$ then $\gamma_{\ddot{d}g}(G) = n$ where $n \geq 2$.

Proof: It is trivial that $\gamma_{\ddot{d}g}(K_n) = n$.
Let $G = (K_n - \{e\})$ where $e = uv$. Then u and v are adjacent to all the vertices of the clique K_{n-2} where the vertices u and v are the extreme vertices and therefore they belong to the doubly geodetic set. On the other hand every vertex of $K_n - \{e\}$ is an exreme vertex. Therefore they belong to the doubly geodetic set. This implies that $\gamma_{\ddot{d}g}(G) = n$.

Theorem 3.5. For a connected graph G of order $n \geq 2$, the doubly geodetic domination number $\gamma_{\ddot{d}g}(G) = 2$ if and if only there exist a doubly geodetic set $S = \{x, y\}$ where $d(x, y) \leq 3$

Proof: Consider a graph G constructed with $\{x, y\}$ as a geodetic set. Let (x, u_1, u_2, y) be a $(x - y)$ geodesic and for $(x - y)$ geodesic there exists a distinct geodesic (x, v_1, v_2, y) with (u_1, v_2) and $(u_2, v_1) \in E(G)$. Any graph of this form satisfies the condition of the theorem.

Theorem 3.6: For a connected G of order $n \geq 2$, the doubly geodetic domination number $\gamma_{\ddot{d}g}(G) = 3$ if and only if there exist a doubly geodetic set $S = \{x, y\}$ where $d(x, y) \leq 4$

Lemma 3.7: For a connected graph G with $\gamma(G) = 1$ then $\gamma_{\ddot{d}g}(G) = \ddot{d}g(G)$

Proof: For K_n $\gamma(K_n) = 1$ and also $\gamma_{\ddot{d}g}(K_n) = \text{n} = \ddot{d}g(G)$. Hence $G = K_n$ turns out to be a trivial case.
Now consider the case when $G \neq K_n$ and $\gamma(G) = 1$. As a single vertex domintes all the other vertices of G, it follows that the maximum degree of G is $(n - 1)$ and $diam(G) \leq 2$. As $G \neq K_n$ there will be atleast two vertices in G that are not adjacent and hence $diam(G) = 2$. Let S be a minimum doubly geodetic set of G. Since $G \neq K_n$ there exists a vertex u such that $u \in V(G)$ and $u \notin S$. Given S is a doubly geodetic set,there exists two distinct geodesics say $(u_1 - u_2)$ and $(v_1 - v_2)$ such that it belongs to both of these geodesics where each $\{u_1, u_2, v_1, v_2\} \in S$ but $diam(G) = 2$. This implies that the geodesics $(u_1 - u_2)$ and $(v_1 - v_2)$ are paths containg u respectively. The path could be $u_1 u v_2$ or $v_1 u v_2$ respectively. It is evident that both u_1 and v_1 dominates u and S now become a doubly geodetic dominating set of G. Thus $\gamma_{\ddot{d}g}(G) \leq |S| = \ddot{d}g(G)$ and also $\ddot{d}g(G) \leq \gamma_{\ddot{d}g}(G)$ and this implies the lemma.

Preposition 3.8: For a connected graph G with order n ≥ 2, $\gamma_{\ddot{d}g}(G) \leq n - \left\lfloor\frac{2diam(G)}{3}\right\rfloor$

Proof: Let the diameter of G be $diam(G) = 3p + q$ where p and q are integers with o$\leq q \leq 2$. Also let v_0, v_d be any two vertices in d such that $d(v_0, v_d) = d$ and P be a shortest path between v_0 and v_d. Supose $A = \{v_0, v_3, .., v_{3p}, v_{3p+q}\}$ and $D = V(G) - (V(P) \backslash A)$ forma a doubly dominating set for G. When

$q = 0, |A| = p + 1$ and $|A| = p + 2$ otherwise,and this implies that $|V(P)\backslash A| = \left\lfloor\frac{6p+2q}{3}\right\rfloor = \left\lfloor\frac{2diam(G)}{3}\right\rfloor$ and hence , $\gamma_{\ddot{d}g}(G) \leq n - \left\lfloor\frac{2diam(G)}{3}\right\rfloor$.

Realization Results:
Theorem 4.1: Given a graph G of order n, diameter d and $(n-d) \geq 2$, there exists a graph G with order n and diameter d such that $\gamma_g(G)$=$\gamma_{\ddot{d}g}(G)$.
Proof: G is constructed as follows. Let $P = v_1, v_2, \dots, v_d$ be a path. The graph G can be constructed by adding $(n-d)$ new vertices to P and joining it with v_1. The new graph is of order n with $(n-d+1)$ leaves and for this graph $\gamma_g(G)$=$\gamma_{\ddot{d}g}(G)$.

Theorem 4.2: Let a and n with positive integers such that $2 \leq a \leq n$, then there exists a connected graph such that $\gamma_{\ddot{d}g}(G)$= a and $|V(G)| = n$.
Proof: The result is trivial for $n = 2$ or 3 For $n = 2$, $G = P_2$ and for $n = 3$, $G \in \{P_3, K_3\}$. Consider the case where $n \geq 4$. For $a = n, G$ can be K_n and for $a = (n-1)$ G can be $K_{1,n-1}$. Suppose $a \leq n-2$, let $K_{1,n-2}$. has leaves $\{u_1, u_2, \dots, u_{n-2}\}$. Then G is obtained by adding a new vertex v to the leaves u_i (a$\leq i \leq n-2$). Then the set S = $\{u, u_1, u_2, \dots, u_{n-2}, v\}$ is a minimum doubly geodetic dominating set.

Theorem 4.3: For any two integers $a, b \geq 2$ there exists a connected graph G such that $\gamma(G) = a, \ddot{d}g(G) = b$ and $\gamma_{\ddot{d}g}(G) = a + b$
Proof: Let $a, b \geq 2$ and let H be the graph constructed by taking a copy of C_6 with antipodal vertices u and v and by adding new vertices $u_1, u_2, \dots, u_{b-1}$ each of them is joined to u. Let G be a graph obtained from H by taking a copy of path on $3(a-2)+1$ vertices $v_0 v_1 \dots . v_{3(a-2)}$ and joining v_0 to the vertex v. The sets $(u, v, v_2, v_5, \dots ., v_{3(a-2)-1})$ and $(u_1, u_2, \dots, u_{b-1}, v_{3(a-2)})$ are a minimum dominating set and a minimum doubly geodetic set of G respectively. Thus $\gamma(G) = a, \ddot{d}g(G) = b$. Also the union of these two set give the minium doubly geodetic dominating set.

Complexity of the Doubly Geodetic Domination Problem:
Theorem 5.1: The doubly geodectic domination problem is NP- complete. The dominating set problem is a well known NP-complete problem and the proof for the NP-completeness of the doubly geodetic domination problem can be derived from the same.

The graph $\bar{G}(\bar{V}, \bar{E})$ can be constructed from $G(V, E)$ as follows. The vertex set $\bar{V}$ is $\bar{V} = V \cup V' \cup V''$ where the vertex set V' induces a clique and V'' induces an independent set. The edge set of G is $\bar{E}$=E$\cup E' \cup E''$. The vertex set V' along with the edge set E' forms a complete graph. The edge set E'' is given by $E'' = \{vv'\} \cup \{v'v_1''\} \cup \{v'v_2''\}, v \in V$

The $\bar{G}$ is composed of three layers , the top layer consisit of G itself , while the middle layer forms a clique of order $|V|$ and the bottom layer consist of independent set of order $2|V|$. It is clear that if X is a doubly geodetic dominating set of $\bar{G}$ then there exsist a doubly geodetic dominating set Y where $|Y| \leq |X|$ such that $Y = D \cup V''$ and $D \subseteq V$. Let D be a dominating set of G and for the given vertex set $D \cup V''$ in $\bar{G}$ the path sets can be defined as follows : $\breve{Y}$=$\{uvv'v_i'' : u \in D, uv \in E, i = 1 \text{ or } 2\}$, $\bar{Z} = \{x_i''x'y'y_i'' : x'', y'' \in V'', i = 1 \text{ or } 2\}$ and $\overline{W} = \{uu'v'v_i'' : u \in D; i = 1, 2; v_i'' \in V''\}$. Each path of $\breve{Y}$ is a geodesic . The geodesics from $\breve{Y}, \bar{Z}$ and $\overline{W}$ covers twice all the vertices of $\bar{G}$. Any $\tilde{I}$(D$\cup V''$) that contains $\breve{Y} \cup \bar{Z} \cup \overline{W}$ is adoubly geodetic dominating set of $\bar{G}$.

Conversly, let $D \cup V''$ is a doubly geodetic dominating set of $\bar{G}$ and $u \in D$,(D$\cup V''$) will contain the geodesic $uvv'v_i''$ which covers the vertex $v \in V\backslash S$ twice.Since $D \cup V''$ is a doubly geodetic dominating set of $\bar{G}$, there will be atleast one vertex in D that is adjacent to v, otherwise v will not be doubly geodominated by $\tilde{I}(D \cup V'')$.Hence D is dominating set of G.

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