DISCRETE HEAT EQUATION MODEL FOR ROD BY PARTIAL FIBONACCI DIFFERENCE OPERATOR

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Abstract: Partial difference equations have extended applications in heat equations. Having introduced the Fibonacci difference equation in the partial section by using Fibonacci difference operators with shift values, a model for heat transfer in the rod is found. Having recourse to Fourier law of cooling, an involved study is carried out to evaluate the movement of heat and thus numerous solutions are postulated. The results obtained are validated by MATLAB and applications are derived.

Keywords: Partial Difference Equation, Partial Difference Operator And Discrete Heat Equation.

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1. Introduction: The difference operator $\underset{\alpha}{\Delta}$ defined on $u(k)$, introduced by Jerzy Popenda [1,4,5] in 1984 found its extension in 2011 when $\underset{\alpha(\ell)}{\Delta}$ was introduced by M.M.S.Manuel, et.al, [6]. The operator was further extended by G.B.A. Xavier, et.al,[2,3] with the introduction of k-difference operator $\underset{k(l)}{\Delta}$ with variable coefficients as $\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k)$. Here, we extend the theory of α-difference operator Δ_α to Fibonacci difference operator $\underset{x(\ell)}{\Delta}$, where $x = (x_1, x_2, \ldots, x_n)$ and $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ and obtain heat equation model by this Fibonacci difference operator.

2. Fibonacci Difference Operator On r –Variables: For $x = (x_1, x_2, x_3, \ldots, x_r)$, the x-difference operator on r-variables real valued function with shift values $\ell = (\ell_1, \ell_2)$ is defined as,

$$\underset{x(\ell)}{\Delta} v(k_1,k_2) = v(k_1,k_2) - x_1 v(k_1-\ell_1, k_2-\ell_2) - x_2 v(k_1-2\ell_1, k_2-2\ell_2) - x_3 v(k_1-3\ell_1, k_2-3\ell_2) - \ldots - x_r v(k_1-r\ell_1, k_2-r\ell_2) \quad (1)$$

The operator in (1) becomes partial Fibonacci difference operator if either ℓ_1 or ℓ_2 is zero but not both. The equations involving a first order linear partial x-difference equation is,

$$\underset{x(\ell)}{\Delta} v(k) = u(k), \ell = (0,\ell_2) \quad \text{or} \quad (\ell_1, 0); \quad x = (x_1, x_2, \ldots, x_r) \quad (2)$$

The equation (2) has a summation form solution

$$v(k_1,k_2) - \sum_{i=0}^{m} \sum_{j=i}^{m} x_j F_{n+i-j} v(k_1, k_2-(n+i)\ell_2) = \sum_{i=0}^{n} F_i u(k_1, k_2 - i\ell_2) \quad (3)$$

where $F_0 = 1$, $F_1 = x_1$, $F_2 = x_1F_0 + x_0F_1$,…, $F_r = x_rF_0 + x_{r-1}F_1 + \ldots + x_1F_{r-1}$, $F_n = x_rF_{n-r} + x_{r-1}F_{n-r+1} + \ldots + x_1F_{n-1}$ for n > r and $F_n = 0$, when $n < 0$ are the rth order Fibonacci numbers.

3. Discrete Heat Equation of A Long Rod: If the temperature of the long rod with position k_1 and time k_2 of long rod is $v(k_1,k_2)$ [3], by (1) and Fourier's cooling law, the discrete heat equation of rod is expressed as

$$\underset{x(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma \underset{x(\pm\ell_1,0)}{\Delta} v(k_1,k_2); \quad x = (x_1,x_2,\ldots,x_r) \qquad (4)$$

where $\underset{x(\pm\ell_1,0)}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta}$. Our main aim of this paper is to study and discuss the solution of the heat equation (4).

Theorem 3.1 *If* $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2) = \underset{x(\pm\ell_1)}{u}(k_1,k_2)$ *are known, then equation* (4) *has a solution*

$$v(k_1,k_2) = \sum_{i=0}^{m}\sum_{j=i}^{m} x_j F_{n+i-j} v(k_1,k_2-(n+i)\ell_2) + \gamma \sum_{i=0}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1,k_2-i\ell_2) \qquad (5)$$

Proof: By representing $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2) = \underset{x(\pm\ell_1)}{u}(k_1,k_2)$, (4) becomes

$$v(k_1,k_2) = \sum_{i=0}^{m}\sum_{j=i}^{m} x_j F_{n+i-j} v(k_1,k_2-(n+i)\ell_2) + \gamma \underset{x(0,\ell_2)}{\overset{-1}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2) \qquad (6)$$

The proof of (5) follows from the relation

$\underset{x(0,\ell_2)}{\overset{-1}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2) = \sum_{i=1}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1-r(0),k_2-i\ell_2)$ and (6).

Theorem 3.2: *Let* $v(k_1\pm\ell_1,*) = v(k_1+\ell_1,*) + v(k_1-\ell_1,*)$ *and* $v(k_1\pm 2\ell_1,*) = v(k_1+2\ell_1,*) + v(k_1-2\ell_1,*)$. *Then the following solutions are similar.*

(a). $v(k_1,k_2) = \frac{{x_1}^n}{(1-2\gamma)^n} v(k_1,k_2-n\ell_2) - \sum_{i=1}^{n} \frac{\gamma {x_1}^i}{(1-2\gamma)^i} v(k_1\pm\ell_1,k_2-(i-1)\ell_2)$

$+\sum_{r=2}^{n} \Big\{\sum_{i=1}^{r} \frac{x_r {x_1}^{i-1}}{(1-2\gamma)^i}[v(k_1,k_2-(i+(r-1))\ell_2)$

$-\gamma v(k_1\pm r\ell_1,k_2-(i-1)\ell_2)]\Big\}$,(7)

(b). $v(k_1,k_2) = \frac{(1-2\gamma)^n}{{x_1}^n} v(k_1,k_2+n\ell_2) + \sum_{i=1}^{n} \frac{\gamma(1-2\gamma)^{i-1}}{{x_1}^{i-1}} v(k_1\pm\ell_1,k_2+i\ell_2)$

$-\sum_{r=2}^{n} \Big\{\sum_{i=1}^{r} \frac{x_r(1-2\gamma)^{i-1}}{{x_1}^i}[v(k_1,k_2+(i-r)\ell_2)$

$-\gamma v(k_1\pm r\ell_1,k_2+i\ell_2)]\Big\}$,(8)

(c). $v(k_1,k_2) = \frac{1}{\gamma^n} v(k_1-n\ell_1,k_2-n\ell_2) - \sum_{i=0}^{m} \frac{1}{\gamma^{i-1}} v(k_1-(i+1)\ell_1,k_2-(i-1)\ell_2)$

$-\sum_{i=0}^{m} \frac{(1-2\gamma)}{x_1\gamma^i} v(k_1-i\ell_1,k_2-(i-1)\ell_2)$

$-\sum_{r=2}^{n} \Big\{\sum_{i=1}^{r} \frac{x_r}{x_1\gamma^{i-1}}[v(k_1-(i-r)\ell_1,k_2-(i-1)\ell_2)$

$+v(k_1-(i+r)\ell_1,k_2-(i-1)\ell_2)]\Big\}$

$$+\sum_{r=2}^{n}\Big\{\sum_{i=1}^{r} \frac{x_r}{x_1\gamma^i} v(k_1-i\ell_1,k_2-(i+(r-1))\ell_2)\Big\}, \qquad (9)$$

(d). $v(k_1,k_2) = \frac{1}{\gamma^n} v(k_1+n\ell_1,k_2-n\ell_2) - \sum_{i=0}^{m} \frac{1}{\gamma^{i-1}} v(k_1+(i+1)\ell_1,k_2-(i-1)\ell_2)$

$-\sum_{i=0}^{m} \frac{(1-2\gamma)}{x_1\gamma^i} v(k_1+i\ell_1,k_2-(i-1)\ell_2)$

$-\sum_{r=2}^{n} \Big\{\sum_{i=1}^{r} \frac{x_r}{x_1\gamma^{i-1}}[v(k_1+(i-r)\ell_1,k_2-(i-1)\ell_2)$

$+v(k_1+(i+r)\ell_1,k_2-(i-1)\ell_2)]\Big\}$

$$+\sum_{r=2}^{n}\Big\{\sum_{i=1}^{r} \frac{x_r}{x_1\gamma^i} v(k_1+i\ell_1,k_2-(i+(r-1))\ell_2)\Big\}. \qquad (10)$$

Proof: (a). From (4), directly generates the relation

$$v(k_1,k_2) = \frac{x_1}{(1-2\gamma)} v(k_1,k_2-\ell_2)+\ldots+\frac{x_r}{(1-2\gamma)} v(k_1,k_2-r\ell_2)$$

$$-\frac{x_1\gamma}{(1-2\gamma)} v(k_1\pm\ell_1,k_2)-\ldots-\frac{x_r\gamma}{(1-2\gamma)} v(k_1\pm r\ell_1,k_2). \qquad (11)$$

By replacing k_2 by $k_2-\ell_2, k_2-2\ell_2,\ldots,k_2-m\ell_2$ in (11), we obtain expressions for $v(k_1,k_2-r\ell_2)$ and $v(k_1\pm r\ell_1,k_2)$.

(b). From (4), we get the following equation

$$v(k_1,k_2)=\frac{(1-2\gamma)}{x_1}v(k_1,k_2+\ell_2)+\gamma v(k_1\pm\ell_1,k_2+\ell_2)$$
$$-\frac{x_2}{x_1}[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm 2\ell_1,k_2+\ell_2)]$$
$$-\ldots-\frac{x_{(r-1)}}{x_1}[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm(r-1)\ell_1,k_2+\ell_2)]$$
$$-\frac{x_r}{x_1}[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm r\ell_1,k_2+\ell_2)] \qquad (\qquad (12)$$

Substituting k_2 by $k_2+\ell_2, k_2+2\ell_2,\ldots,k_2+m\ell_2$ repeatedly, we get the result.

(c). A simple calculation on (4) gives the expression

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1-\ell_1,k_2-\ell_2)-v(k_1-2\ell_1,k_2)+\frac{x_2}{x_1\gamma}v(k_1-\ell_1,k_2-2\ell_2)$$
$$+\frac{x_3}{x_1\gamma}v(k_1-\ell_1,k_2-3\ell_2)+\ldots+\frac{x_r}{x_1\gamma}v(k_1-\ell_1,k_2-r\ell_2)$$
$$-\frac{x_2}{x_1}[v(k_1+\ell_1,k_2)+v(k_1-3\ell_1,k_2)]-\frac{x_3}{x_1}[v(k_1+2\ell_1,k_2)+v(k_1-4\ell_1,k_2)]$$
$$-\ldots-\frac{x_r}{x_1}[v(k_1+(r-1)\ell_1,k_2)+v(k_1-(r+1)\ell_1,k_2)]-\frac{(1-2\gamma)}{x_1\gamma}v(k_1-\ell_1,k_2)$$

By substituting k_1 by $k_1-\ell_1,k_1-2\ell_1,\ldots,k_1-m\ell_1$ and doing the same for k_2 by $k_2-\ell_2,k_2-2\ell_2,\ldots,k_2-m\ell_2$ repeatedly.

(d). (4) gives the expression

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1+\ell_1,k_2-\ell_2)-v(k_1+2\ell_1,k_2)+\frac{x_2}{x_1\gamma}v(k_1+\ell_1,k_2-2\ell_2)$$
$$+\frac{x_3}{x_1\gamma}v(k_1+\ell_1,k_2-3\ell_2)+\ldots+\frac{x_r}{x_1\gamma}v(k_1+\ell_1,k_2-r\ell_2)$$
$$-\frac{x_2}{x_1}[v(k_1+3\ell_1,k_2)+v(k_1-\ell_1,k_2)]-\frac{x_3}{x_1}[v(k_1+4\ell_1,k_2)+v(k_1-2\ell_1,k_2)]$$
$$-\ldots-\frac{x_r}{x_1}[v(k_1+(r+1)\ell_1,k_2)+v(k_1-(r_1)\ell_1,k_2)]-\frac{(1-2\gamma)}{x_1\gamma}v(k_1+\ell_1,k_2)$$

The proof of (d) results from replacing k_1 by $k_1+\ell_1,k_1+2\ell_1,\ldots,k_1+m\ell_1$ and k_2 by $k_2-\ell_2,k_2-2\ell_2,\ldots,k_2-m\ell_2$ repeatedly.

Example 3.3 *Suppose that* $v(k_1,k_2)=e^{k_1+k_2}$ *is a exact solution of* (4), *then we have the solution* $\underset{0,\ell_2(x)}{\Delta}e^{k_1+k_2}=\gamma[\underset{\ell_1(x)}{\Delta}e^{k_1+k_2}+\underset{-\ell_1(x)}{\Delta}e^{k_1+k_2}]$, which yields

$$e^{k_1+k_2}-x_1e^{k_1+k_2-\ell_2}-\ldots-x_re^{k_1+k_2-r\ell_2}=\gamma[e^{k_1+k_2}-x_1e^{k_1\pm\ell_1+k_2}-\ldots-x_re^{k_1\pm r\ell_1+k_2}]$$

Cancelling $e^{k_1+k_2}$ on both sides derives $\gamma=\frac{1-x_1e^{-\ell_2}-x_2e^{-2\ell_2}}{2-x_1(e^{\ell_1}+e^{-\ell_1})-x_2(e^{2\ell_1}+e^{-2\ell_1})}$ (13)

For numerical verification, we give the MATLAB coding for (a) of Theorem (3.2) when $m=1$, $r=2$, $k_1=1$, $\ell_1=1$, $k_2=2$, $\ell_2=2$, $x_1=1$, $x_2=2$, $v(k_1,k_2)=e^{(k_1+k_2)}$ and γ is as given in (13).

$$((1.\wedge 1)./(1.102638526.\wedge 1)).*exp(3-(1.*2))-(symsum((((-0.051319263).*(1.\wedge i))./(1.102638526.\wedge(i))).*(exp(4-((i-1).*2))+(exp(2-((i-1).*2)))),i,1,1))+(symsum(((2.*(1.\wedge(i-1)))./(1.102638526.\wedge i)).*(exp(3-((i+1).*2))+(0.051319263.*((exp(5-((i-1).*2))+exp(1-((i-1).*2)))))),i,1,1))$$

4. Conclusion: The newly introduced partial difference operator with its corresponding equations has many applications in the field of finite difference methods and heat equations. The nature of propagation of heat through the rod is studied using partial Fibonacci difference operator. The results obtained above gives us the tool to predict the temperature and also gives us the possibility to determine the nature of the rod under study for better transmission.

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