ON $T_i^{I^{(T)\beta}}(i = 0, 1, 2)$-SPACES IN INTUITIONISTIC TOPOLOGICAL SPACES

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Abstract: In this paper, I introduce spaces called a $T_i^{I^{(T)\beta}}(i = 0,1,2)$-Spaces in intuitionistic topological spaces. Also discuss some of their properties.

Keywords: $T_i^{I^{(T)\beta}}(i = 0, 1, 2)$-Spaces.

1. Introduction: After Atanassov [2] introduced the concept of intuitionistic fuzzy sets" as a generalization of fuzzy sets, it becomes a popular topic of investigation in the fuzzy set community. Many mathematical advantages of intuitionistic fuzzy sets are discussed in [2, 3, 4]. Coker generalized topological structures in fuzzy topological spaces to intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. Later many researchers have studied topics related to intuitionistic fuzzy topological spaces. Coker [11] also introduced the concept of intuitionistic topological spaces with intuitionistic sets, and investigated basic properties of continuous functions and compactness. He and his colleague [12, 13, 14] also examined separation axioms in intuitionistic topological spaces.

2. Preliminaries:

Definition 2.4: [11, 13] An intuitionistic topology (IT for short) on a nonempty set X is a family$T$ of $IS's$ in X satisfying thefollowing axioms:
$(i)\phi_\sim, X_\sim \in T$,
$(ii)G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,
$(iii) \cup G_i \in T$ for any arbitrary family $\{G_i : i \in J\} \subseteq T$.

In this case the pair (X, T) is called an intuitionistic topological space (ITS for short) and any IS in T is known as an intuitionistic open set (IOS for short) in X.

Example 2.5: [13] Any topological space (X, T) is obviously an ITS in the form $T = \langle A', A \in T_0\rangle$, whenever we identify a subset A in X with its counterpart $A' = \langle X, A, A^c\rangle$ as before.

Example 2.6: [11] Let $X = \{a, b, c, d, e\}$ and consider the family $T = \{\phi_\sim, X_\sim, G_1, G_2, G_3, G_4\}$ where $G_1 = \langle X, \{a, b\}, \{d\}\rangle$, $G_2 = \langle X, \{a, c, e\}, \{b, d\}\rangle$, $G_3 = \langle X, \{a\}, \{b, d, e\}\rangle$and$G_4 = \langle X, \{a, d\}, \{e\}\rangle$. Then (X, T) is an ITS on X.

Example 2.7: [11] Let $X = \{a, b, c, d, e\}$ and consider the family $T = \{\emptyset_\sim, X_\sim, A_1, A_2, A_3, A_4\}$, where $A_1 = \langle X, \{a, b, c\}, \{d\}\rangle$, $A_2 = \langle X, \{c, d\}, \{e\}\rangle$, $A_3 = \langle X, \{c\}, \{d, e\}\rangle$ and $A_4 = \langle X, \{a, b, c, d\}, \emptyset\rangle$. Then (X, T) is an ITS on X.

Definition 2.8: [11] The complement A of an $IOSA$ in an ITS (X, T) is called an intuitionistic closed set (ICS for short) in X.
Now we define closure and interior operations in $ITS's$:

Definition 2.9: [11] Let (X, T) be an ITS and $A = \langle X, A^1, A^2\rangle$ be an IS in X. Then the interior and closure of A are defined by

$$int(A) = \cup\{K: K \text{ is an } IOS \text{ in } X \text{ and } K \subseteq A\},$$
$$cl(A) = \cap\{K: K \text{ is an } ICS \text{ in } X \text{ and } A \subseteq K\},$$

In this paper we have used $I^{(T)}c(A)$ instead of $cl(A)$ and $I^{(T)}i(A)$ instead of $int(A)$.
It can be also shown that $I^{(T)}c(A)$ is an ICS and $I^{(T)}i(A)$ is an IOS in X, and A is an ICS in Xiff$I^{(T)}c(A) = A$ and A is an IOS in X iff$I^{(T)}i(A) = A$.

Example 2.10: [11] Let $X = \{a, b, c, d, e\}$ and consider the family $T = \{\phi_{\sim}, X_{\sim}, G_1, G_2, G_3, G_4\}$ where $G_1 = \langle X, \{a, b, c\}, \{d\}\rangle$, $G_2 = \langle X, \{c, d\}, \{e\}\rangle$, $G_3 = \langle X, \{c\}, \{d, e\}\rangle$ and $G_4 = \langle X, \{a, b, c, d\}, \{\phi\}\rangle$. If $B = \langle X, \{a, c\}, \{d\}\rangle$, then we can write down $I^{(T)}i(B) = \langle X, \{c\}, \{d, e\}\rangle$ and $I^{(T)}c(B) = \langle X, X, \phi\rangle = X_{\sim}$.

Proposition 2.11: [11] For any ISA in (X, T) we have $I^{(T)}c(\bar{A}) = \overline{I^{(T)}\iota(A)}$ and$I^{(T)}i(\bar{A}) = \overline{I^{(T)}c(A)}$.

Proposition 2.12: [11] Let (X, T) be an ITS and A, B be IS's in X. Then the following properties hold:
(a) $int(A) \subseteq A$; (b) $A \subseteq cl(A)$.
(c) $A \subseteq B \Rightarrow int(A) \subseteq int(B)$; (d) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$;
(e) $int(int(A)) = int(A)$; (f) $cl(cl(A)) = cl(A)$;
(g) $int(A \cap B) = int(A) \cap int(B)$; (h) $cl(A \cup B) = cl(A) \cup cl(B)$;
(i) $int(X_{\sim}) = X_{\sim}$; (j) $cl(\phi_{\sim}) = \phi_{\sim}$.

Definition 2.13: [P. Basker et al.] [5]An intuitionistic set $A = \langle X, A^1, A^2\rangle$ in an $ITS(X, T)$ is said to be
(a) intuitionistic semi-preopen [briefly. $I^{(T)}SP$-open (or) $I^{(T)}\beta$-open]if there exists $B \in I^{(T)}PO(X)$ such that $B \subseteq A \subseteq I^{(T)}c(A)$.
(b) intuitionistic semi-preclosed [briefly. $I^{(T)}SP$-closed (or) $I^{(T)}\beta$-closed]if there exists an intuitionistic preclosed set B such that $I^{(T)}i(A) \subseteq A \subseteq B$.

The family of all $I^{(T)}\beta$-open [resp. $I^{(T)}\beta$-closed] sets of an $ITS(X, T)$ will be denoted by $I^{(T)}\beta O(X)$ [resp. $I^{(T)}\beta C(X)$].

Example 2.14: [P. Basker et al.] [5] Let $X = \{a_1, a_2, a_3\}$ and
$G_1 = \langle X, \{a_1\}, \{a_2, a_3\}\rangle$,
$G_2 = \langle X, \{a_1, a_2\}, \{a_3\}\rangle$,
$G_3 = \langle X, \{a_1\}, \{a_2\}\rangle$.
Then $T = \{\phi_{\sim}, X_{\sim}, G_1, G_2\}$ is an intuitionistic topology on X and $G_3 \in I^{(T)}PO(X)$. Let $A = \langle X, \{a_1, a_3\}, \{a_2\}\rangle$ be an IS in (X, T). Then $G_3 \subseteq A \subseteq I^{(T)}c(G_3)$, and hence $A \in I^{(T)}\beta O(X)$.

Definition 2.15: [P. Basker et al.] [5] Let (X, T) be an intuitionistic topological space and A be a subset X. Then:
$(a)$$I^{(T)}\beta$-interior of A is the union of all $I^{(T)}\beta$-open sets contained in A and it is denoted by $Int_{I^{(T)}}\beta(A)$. $Int_{I^{(T)}}\beta(A) = \cup\{U: U \text{ is an } I^{(T)}\beta - \text{open set and } U \subseteq A\}$.
$(b)$$I^{(T)}\beta$-closure of A is the intersection of all $I^{(T)}\beta$-closed sets containing A and it is denoted by $Cl_{I^{(T)}}\beta(A)$. $Cl_{I^{(T)}}\beta(A) = \cap\{U: U \text{ is an } I^{(T)}\beta - \text{closed set and } A \subseteq U\}$.

3. *On* $T_i^{I^{(T)}\beta}$ $i = 0, 1, 2$-*Spaces*

Definition 3.1: An intuitionistic topological space (X, T) is said to be
$T_0^{I^{(T)}}\beta$ if for each pair of distinct points x, y in X, there exists a $I^{(T)}\beta$-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.
$T_1^{I^{(T)}}\beta$ if for each pair of distinct points x, y in X, there exist two $I^{(T)}\beta$-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.
$T_2^{I^{(T)}}\beta$ if for each distinct points x, y in X, there exist two disjoint $I^{(T)}\beta$-open sets U and V containing x and y respectively.

Definition 3.2: (a) Let (X, T) be an intuitionistic topological space and let $x \in X$. A subset N of X is said to be $I^{(T)}\beta$-nbhd of x if there exists an $I^{(T)}\beta$-open set G such that $x \in G \subset N$.

The collection of all $I^{(T)}\beta$-nbhd of $x \in X$ is called an $I^{(T)}\beta$-nbhd system at x and shall be denoted by $n^{\#}_{I^{(T)}\beta}(x)$.
(b) Let (X,T) be an intuitionistic topological space and A be a subset of X, A subset N of X is said to be $I^{(T)}\beta$-nbhd of A if there exists an $I^{(T)}\beta$-open set G such that $A \in\ G \subset N$.
(c) Let A be a subset of X. A point $x \in A$ is said to be an $I^{(T)}\beta$-interior point of A, if A is an $n^{\#}_{I^{(T)}\beta}(x)$. The set of all $I^{(T)}\beta$-interior points of A is called an $I^{(T)}\beta$-interior of A and is denoted by $I_{I^{(T)}\beta}(A)$.

Proposition 3.3: An intuitionistic topological space (X,T) is $T_0^{I^{(T)}}\beta$ if and only if for each pair of distinct points $x,\ y$ of X, $Cl_{I^{(T)}}\beta(\{x\}) \neq Cl_{I^{(T)}}\beta(\{y\})$.
Proof: Necessity. Let (X,T) be a $T_0^{I^{(T)}}\beta$-space and x,y be any two distinct points of X. There exists a $I^{(T)}\beta$-open set U containing x or y, say x but not y. Then X/U is a $I^{(T)}\beta$-closed set which does not contain x but contains y. Since $Cl_{I^{(T)}}\beta(\{y\})$ is the smallest $I^{(T)}\beta$-closed set containing y, $Cl_{I^{(T)}}\beta(\{y\}) \subseteq X/U$ and therefore $x \in Cl_{I^{(T)}}\beta(\{y\})$. Consequently $Cl_{I^{(T)}}\beta(\{x\}) \neq Cl_{I^{(T)}}\beta(\{y\})$.
Sufficiency. Suppose that $x,y \in\ X$, $x \neq y$ and $Cl_{I^{(T)}}\beta(\{x\}) \neq Cl_{I^{(T)}}\beta(\{y\})$. Let z be a point of X such that $z \in Cl_{I^{(T)}}\beta(\{x\})$ but $z \notin Cl_{I^{(T)}}\beta(\{y\})$. We claim that $x \notin Cl_{I^{(T)}}\beta(\{y\})$. For, if $x \in Cl_{I^{(T)}}\beta(\{y\})$ then $Cl_{I^{(T)}}\beta(\{x\}) \subseteq Cl_{I^{(T)}}\beta(\{y\})$. This contradicts the fact that $z \notin Cl_{I^{(T)}}\beta(\{y\})$. Consequently \$x\$ belongs to the $I^{(T)}\beta$-open set $X/Cl_{I^{(T)}}\beta(\{y\})$ to which y does not belong.

Proposition 3.4: An intuitionistic topological space (X,T) is $T_1^{I^{(T)}}\beta$ if and only if the singletons are $I^{(T)}\beta$-closed sets.
Proof: Let (X,T) be $T_1^{I^{(T)}}\beta$ and x any point of X. Suppose $y \in\ X\ /\ \{x\}$ then $x \neq y$ and so there exists a $I^{(T)}\beta$-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subseteq\ X/\{x\}$, that is $X/\{x\} = \cup\ \{U\colon y \in X/\{x\}\}$ which is $I^{(T)}\beta$-open.
Conversely, suppose $\{p\}$ is $I^{(T)}\beta$-closed for every $p \in X$. Let $x,y \in\ X$ with $x \neq y$. Now $x \neq y$ implies $y \in X\ /\ \{x\}$. Hence $X/\{x\}$ is a $I^{(T)}\beta$-open set contains y but not x. Similarly $X/\{y\}$ is a $I^{(T)}\beta$-open set contains x but not y. Accordingly X is a $T_1^{I^{(T)}}\beta$-space.

Theorem 3.5: Let A and B be subsets of X. Then
(a) $Int_{I^{(T)}}\beta(X) = X$ and $Int_{I^{(T)}}\beta(\phi) = \phi$,
(b) $Int_{I^{(T)}}\beta(A) \subset A$,
(c) If B is any $I^{(T)}\beta$-open set contained in A, then $B \subset Int_{I^{(T)}}\beta(A)$,
(d) If $A \subset B$, then $Int_{I^{(T)}}\beta(A) \subset Int_{I^{(T)}}\beta(B)$.
Proof:
(a) Since X and ϕ are $I^{(T)}\beta$-open sets, $Int_{I^{(T)}}\beta(A) = \cup\{G\colon G\ \ is\ I^{(T)}\beta - open, G \subset X\} = X\cup\{all\ \ I^{(T)}\beta - open\ \ sets\} = X$.
That is $Int_{I^{(T)}}\beta(X) = X$. Since ϕ is the only $I^{(T)}\beta$ -open set contained in ϕ , $Int_{I^{(T)}}\beta(\phi) = \phi$.
(b) Let $x \in Int_{I^{(T)}}\beta(A) \Rightarrow x$ is an $I^{(T)}\beta$-interior point of A.
$\Rightarrow A$ is an $n^{\#}_{I^{(T)}\beta}(x)$.
$\Rightarrow x \in A$.
Thus $\in\ Int_{I^{(T)}}\beta(A) \Rightarrow x \in A$. Hence $Int_{I^{(T)}}\beta(A) \subset A$.
(c) Let B be any $I^{(T)}\beta$-open set such that $B \subset A$. Let $x \in B$, since B is an $I^{(T)}\beta$-open set contained in A, then x is an $I^{(T)}\beta$-interior point of A. That is $x \in Int_{I^{(T)}}\beta(A)$. Hence $B \subset Int_{I^{(T)}}\beta(A)$.
(d) Let A and B be subsets of X such that $A \subset B$. Let $x \in Int_{I^{(T)}}\beta(A)$. Then x is an $I^{(T)}\beta$-interior point of A and so A is an $n^{\#}_{I^{(T)}\beta}(x)$. Since $B \supset A$, B is also an $n^{\#}_{I^{(T)}\beta}(x)$. This implies that $x \in Int_{I^{(T)}}\beta(B)$. Thus we have shown that $x \in Int_{I^{(T)}}\beta(A) \Rightarrow x \in Int_{I^{(T)}}\beta(B)$. Hence $Int_{I^{(T)}}\beta(A) \subset Int_{I^{(T)}}\beta(B)$.

Proposition 3.6: The following statements are equivalent for an intuitionistic topological space (X,T):
(a) X is $T_2^{I^{(T)}}\beta$.
(b) Let $x \in X$. For each $y \neq x$, there exists a $I^{(T)}\beta$-open set U containing x such that $y \notin Cl_{I^{(T)}}\beta(U)$.

(c) For each $x \in X$, $\cap \{Cl_{I^{(T)}}\beta(U): U \in I^{(T)}\beta O(X)$ *and* $x \in U\} = \{x\}$.

Proof:

$(a) \Rightarrow (b)$ Since X is $T_2^{I^{(T)}}\beta$, there exist disjoint $I^{(T)}\beta$-open sets U and V containing x and y respectively. So, $U \subseteq X/V$. Therefore, $Cl_{I^{(T)}}\beta(U) \subseteq X/V$. So $y \notin Cl_{I^{(T)}}\beta(U)$.

$(b) \Rightarrow (c)$ If possible for some $y \neq x$, we have $y \in Cl_{I^{(T)}}\beta(U)$ for every $I^{(T)}\beta$-open set U containing x, which then contradicts (b).

$(c) \Rightarrow (a)$ Let $x,\ y \in X$ and $x \neq y$. Then there exists a $I^{(T)}\beta$-open set U containing x such that $y \notin Cl_{I^{(T)}}\beta(U)$. Let $V = X/Cl_{I^{(T)}}\beta(U)$, then $y \in V$ and $x \in U$ and also $U \cap V = \phi$.

Theorem 3.7: If A and B are subsets of X, then $Int_{I^{(T)}}\beta(A) \cup Int_{I^{(T)}}\beta(B) \subset Int_{I^{(T)}}\beta(A \cup B)$

Proof: We know that $A \subset A \cup B$ and $B \subset A \cup B$, $Int_{I^{(T)}}\beta(A) \subset Int_{I^{(T)}}\beta(A \cup B)$ and $Int_{I^{(T)}}\beta(B) \subset Int_{I^{(T)}}\beta(A \cup B)$. This implies that $Int_{I^{(T)}}\beta(A) \cup Int_{I^{(T)}}\beta(B) \subset Int_{I^{(T)}}\beta(A \cup B)$.

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