A STUDY ON SLOPE NUMBER OF CYCLE RELATED GRAPHS

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Abstract: Slope number of a graph G is the minimum number of slopes required to draw a graph G. By using the slopes the drawing of G is obtained by placing the vertices of a regular polygon. Motivated by this fact, cycle related graphs have been chosen for investigation because nowadays cycle related graphs have been the major focus of attention. In this paper, the slope number of cycle related graphs such as tadpole graph, lollipop graph, barbell graph is discussed. Results on characterization, perfect matching and diameter are also analyzed.

Keywords: Slope Number, Tadpole Graph, Barbell Graph, Lollipop Graph, Diameter, Matching.

Introduction: Graph theory is a fascinating playground for exploring the proof techniques in discrete mathematics and its results have applications in many areas of the computing, social, and natural sciences. Also, graph drawing is a familiar concept of graph theory and it has many quality measures and one among them is a slope number. The problem of determining the slope number of a graph was first introduced by Wade and Chu in 1994 [1]. They proved that the slope number of complete graph of k_n is n and they have explored an algorithm to determine whether the complete graph can be drawn with the given set of slopes or not. The slope number problem is an optimization problem and is NP-hard to determine the slope number of any arbitrary graph [2].

A straight line drawing is a drawing of a graph if the vertices of G are represented by distinct points in the plane and each edge is a straight line. A canonical way of drawing of a complete graph is an existing one. The study of straight line drawing using few slopes is related to the study of the slope number. The slope of an edge in a straight line drawing is the family of all straight lines parallel to this edge[3]. Drawings of graphs which use only the horizontal and vertical slopes are called orthogonal. In view of this, the slope number on cycle related graphs are concentrated.

In the present paper, we consider cycle related graphs such as barbell graph, lollipop graph and tadpole graph for investigation. Albert William et al. have shown results on packing chromatic number for some classes of graphs [4]. Ambrus et al. discussed the slope parameter of graphs and showed results on slope number of cycle graph of odd length and even length [5]. In view of this, we preferred tadpole graph, since the graph is obtained from a cycle and a path. The slope number of tadpole graph is determined and characterized.

The slope number on complete graphs is studied earlier [1]. In line of thought, the slope number of barbell graph and lollipop graph is investigated, since the barbell graph is obtained from n copies of complete graph by a bridge and lollipop graph is obtained from complete graph and a path. The slope number of these graphs is studied and its characterization is discussed according to the edges. Next, the perfect matching of these graphs is focussed and examined for odd and even number of vertices. Finally, the diameter of these graphs is figured and the results on slope number of these graphs are expressed in terms of diameter.

Overview of the paper: Let G be a simple, connected and undirected graph. A graph is a non-empty finite set V(G) of elements called vertices and set E(G) of pairs of vertices called edges. A sub-graph of G is a graph whose set of vertices and set of edges are all subsets of G. A cutvertex is a single vertex whose removal disconnects the graph. A cutedge is a single edge whose removal disconnects the graph. In a graph G, the maximum distance between two vertices of G is called the diameter of G and is denoted by diam(G). A matching in a bipartite graph is a set of the edges chosen in such a way that no two edges share an endpoint. A matching is perfect if all the vertices of G are saturated. *A* maximum matching is a matching of maximum size (maximum number of edges). Graph drawing is the problem of constructing geometric representation of graphs. To obtain a pictorial representation of a graph G, points and lines are used as vertices and edges respectively. In view of this, the slope number of tadpole graph, lollipop graph and barbell graph is characterized. Following this, we obtained results on diameter, matching and perfect matching for the defined graphs.

Slope number: The slope number of a graph G is the minimum number of distinct edge slopes required to draw the graph G. It is denoted by $sl[G]$.
From the figure (1),. Let S_1,S_2,S_3,S_4,S_5,S_6 be the slope of the edges V_8V_9, V_1V_8, V_1V_9, V_2V_9, V_2V_3, V_3V_9, The slope number of the graph is 6.

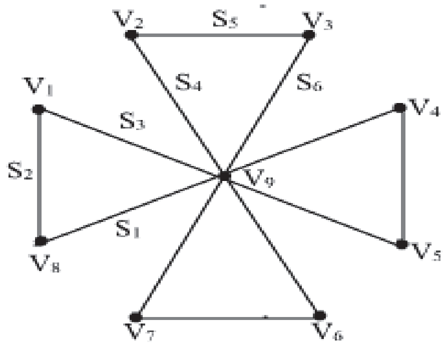

Fig.1 slope number of G is 6

3.1 Theorem: [1] $sl\,[K_m] = \begin{cases} 0, & if\ m = 1 \\ 1, & if\ m = 2 \\ m, & if\ m \geq 3 \end{cases}$

Ambrus et.al, discussed the slope number regarding cycles, according to the parity of the length [4]. For an even cycle, there is a representation with two slopes. This special case has been extended for arbitrary cycles of even length. Also, for odd cycles of length at least 5, an additional slope is used.

3.2. Theorem: [4] $sl\,[C_n] = \begin{cases} 1, & \text{if } n = 3 \\ 2, & \text{if n is even} \\ 3, & \text{if } n \geq 5 \end{cases}$

The main results are as follows.

Slope number of a Tadpole graph: A (m, n) Tadpole graph is the graph obtained by joining a cycle Cm to a path Pn with a bridge. It is denoted by T(m, n). The number of vertices and edges of T(m,n) is m+n. The diameter of T(m,n) is $\left\lfloor \frac{m}{2} \right\rfloor + n$. Figure .2(a) and 2(b) is the illustration of tadpole graph.

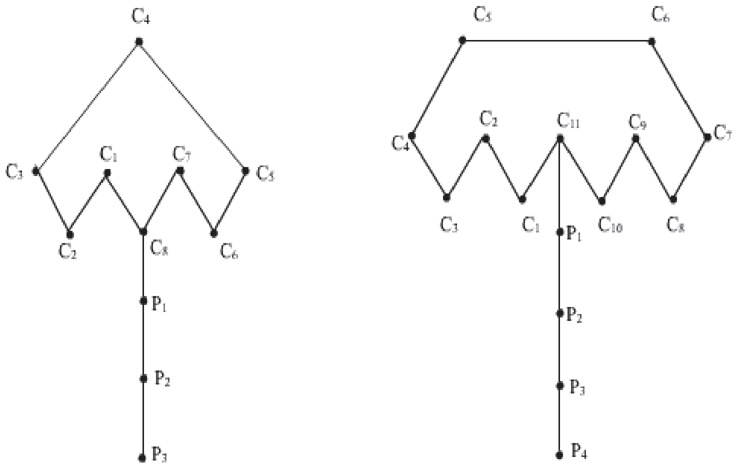

Fig.2(a) T(8,3) Fig.2(b) T(11,4)

4.1 Theorem:
Let $G = T(m,n)$. Then

$$sl[G] = \begin{cases} 3, & if\ n\ is\ even \\ 4, & if\ n\ is\ odd \end{cases}$$

Proof: Let G be a Tadpole graph with the vertex set and edge set.
Case (i) if m is odd, n is even or odd .
Label the vertices of the cycle C as $C_1, C_2, \ldots, C_m$ and vertices of the path as $V_1, V_2, \ldots, V_n$. Let u,v,w be the three consecutive vertices of C. Choose the first vertex of u of C. The edges incident to the vertex u∈C shares atmost two slopes and draw the edges. Let it be S_1 and S_2. Choose the second vertex v∈C and the edges incident to the vertex v shares 2 distinct slopes. But the slope of an edge is already shared by the vertex u and hence the slope of vw is 1. Let it be S_3. After the construction of cycle C, by joining the path graph P_n to the vertex of C_m, the tadpole graph is obtained. Since the line segment of the path graph being horizontal, the slope of the edges of P_n shares the same slope, which is not used by the slopes of C_m. Let it be S_4.

$\therefore sl[G] = sl[C_m] + sl[P_n]$, where m is odd

$= 3+1$

$= 4.$

Case (i) if m is even, n is even or odd.
Let u,v,w be the three consecutive vertices of cycle C. Choose the vertex as discussed in case (i). The edges incident to the vertex v shares 2 distinct slopes. Let it be S_1 and S_2. By similar argument of case (i), the edge of the path graph has the same slope and is denoted by S_3.

$\therefore sl[G] = sl[C_m] + sl[P_n]$, where m is even

$= 2+1$

$= 3.$

Hence the proof.

4.2 Theorem: Let G=T(m,n). Then G has a cut vertex if and only if d(v)=3.

Proof:

Assume that G is a tadpole graph with d(v)=3
To Prove that G has a cut vertex
Let uv∈G be a bridge with one vertex incident on cycle and other vertex incident on path. Since uv is a bridge either u or v will be a cutvertex. If G-u, then G is disconnected.
Hence the proof.

4.3 Theorem: Let G=T(m,n). Then the slope number of G with a cut vertex is bounded by atmost 4, only if m is odd.
4.4 Theorem: Let G=T(m,n). Then the slope number of G with a cut vertex is bounded by atmost 3, if m is even.
4.5 Theorem: Let G=T(m,n). Then the edge e of G is a cutedge can be drawn with 4 slopes if m is odd.
Proof:
By theorem 4.2, If G has a cutvertex, then it has a bridge.
Hence the proof.
4.6 Theorem: Let G=T(m,n). Then the edge e of G is a cutedge can be drawn with 3 slopes if m is even.
4.7 Observation: Let G=T(m,n). Then there exists a perfect matching of G if m is even and n is even.
4.8 Observation: Let G=T(m,n). Then there exist a perfect matching of G if m is odd and n is odd.
4.9 Theorem
Let G=T(m,n). Then $\alpha'(G) = \frac{m+n}{2}$
4.10 Theorem: Let G=T(m,n). Then sl[G]≤ diam[G] for m>3.

Slope number of a Barbell graph: The n-barbell graph is a simple graph obtained by connecting two copies of a complete graph K_m by a bridge and it is denoted by B(K_m ,K_m). The number of vertices and edges of barbell graph is $2m$ and $m(m-1)+1$ respectively. The diameter of B(K_m ,K_m) is 3. Figure 3 is the illustration of barbell graph.

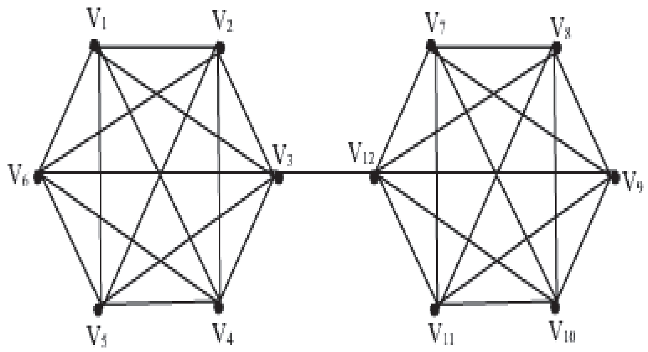

Fig.3 B(K_6,K_6)

5.1 Theorem:
Let G= B (K_m ,K_m). Then sl[G] = m, if the following condition holds.
(i) sl[K_m] = m.
(ii) K_m has atleast one horizontal edge.
Proof:
Let K_m be a complete graph on m vertices.
By theorem, sl[K_m] = m …... con (i).
Now consider 2 copies of complete graph K_m. Sine each K_m is isomorphic to each other it contributes the same slope. By drawing an edge between each copies of K_m, barbell graph is obtained. The slope of an edge connected between each copy of K_m is already used by atleast one vertex of K_m. Therefore by con (ii) K_m has atleast one horizontal edge which shares the same slope of K_m.
i.e ,sl[G] = m
Hence the proof.

5.2 Theorem:
Let G= B (K_m ,K_m). Then the slope of the graph is odd or even if and only if it has odd number of edges.

5.3 Theorem:
Let G = B (K_m ,K_m). Then e is a cutedge of G can be drawn with m slopes.
5.4 Theorem:
Let G = T(m,n). Then v is a cut vertex of G if and only if d(v)=m.
5.6 Theorem: Let G = B(K_m ,K_m). Then there exists K_m and $K_{m-1.}$ Which can be drawn with m and m-1 slopes respectively, iff G has a cutvertex of G.
5.7 Theorem: Let G = B (K_m ,K_m). Then sl[G] = diam[G] + m.
Remark:
Perfect matching does not exist for odd number of vertices of K_m.
5.8 Theorem: Let G = B (K_m ,K_m). Then there exist a perfect matching of G.
5.9 Theorem: Let G = B (K_m,K_m). Then
$\alpha'(G) = \left\{\frac{sl[G]+n}{2}\right.$, if m,n is odd and even.

Slope number of Lollipop graph: A (m,n) Lollipop graph is the graph obtained by joining a complete graph Km to a path Pn with a bridge. It is denoted by L(m, n). The vertices and edges of lollipop graph is $m + n$ and $\frac{m(m-1)}{2} + n$ respectively. The diameter of L(m, n) is n+1. Figure . 4 is the illustration of lolliop graph.

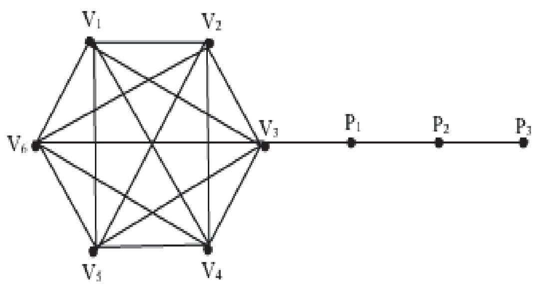

Fig .4 L(6,3)

6.1 Theorem: Let G = L(m,n). Then *sl*[G] = m, if the following conditions hold.
(i) sl[K_m] = m.
(ii) The edges of the path graph shares exactly one slope of K_m
Proof:
Let K_m be a complete graph on m vertices.
By theorem sl[K_m] = m ....con(i)
Now consider the path graph P_n on n vertices. P_n is attached to one end vertex of K_m in drawing of G. Since by condition(ii), the edges on the path graph shares the same slope which is already used by K_m.
i.e *sl*[G] = m
Hence the proof.
6.2 Theorem: Let G=L(m,n), Then the slope of the graph is odd if and only if m is odd and n is odd or even.
6.3 Theorem: Let G=L(m,n). Then the slope of the graph is even if and only if m is even and n is odd or even.
6.4 Theorem: Let G=L(m,n). Then the vertex v is a cutvertex if and only if d(v) = n+1
6.5 Theorem: Let G=L(m,n). Then there exists K_{m-1} and P_n iff G has a cut vertex.
Proof:
Let v be a cut vertex of G, then G-v is disconnected and hence K_{m-1} and P_n are different components of G-v, then there is no edge in G-v.
Hence the proof.
6.6 Theorem: Let G=L(m,n). Then the edge e of G is a cutedge can be drawn with m slopes.

6.7 Observation: Let G=L(m,n). Then there exist a perfect matching if m and n is even.
6.8 Observation: Let G=L(m,n). Then there exist a perfect matching if m and n is odd.
6.9 Theorem: Let G=L(m,n). Then sl[G]=diam[K_m] + n
6.10 Theorem: Let G=L(m,n). Then
$\alpha'(G) = \begin{cases} \frac{sl[G]+n}{2} \end{cases}$, if m,n both odd and even

Conclusion: Thus we determined the slope number of tadpole graph, barbell graph and lollipop graph. We have also characterized the slope number with respect to edges of the defined graphs. Also the perfect matching of these graphs is observed and the diameter is expressed in terms of slope number.

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