WAVELET BASED NUMERICAL METHOD FOR THE SOLUTION OF INTEGRAL EQUATIONS

S. C. Shiralashetti
Professor, Dept. of Mathematics, Karnatak University, Dharwad-03, Karnataka, India
S. S. Naregal
Assistant Teacher, Govt. High School, Devarahuballi, Dharwad-580003, Karnataka, India
S. I. Hanaji
Research Scholar, Dept. of Maths, Karnatak University, Dharwad-03, Karnataka, India
M. S. Gali
Research Scholar, Dept. of Maths, Karnatak University, Dharwad-03, Karnataka, India
manjugalijack@gmail.com.com

Abstract: In this Paper, A wavelet based numerical method for the solution of integral equations is presented. The Cosine and Sine wavelet matrices are utilized to reduce the integral equations into a system of algebraic equations and Cosine and Sine (CAS) wavelet coefficients are computed using Matlab. Proposed method is tested on some illustrative examples and the numerical findings are compared with the exact solutions to show the Accuracy and efficiency of the scheme.

Keywords: Integral equations, Cosine and Sine wavelets, Operational Matrices.

1. Introduction: Integral equations found its applications in several fields of science and engineering. There are some numerical methods for approximating the solution of integral equations of second kind is known and many basis functions have been used [1 & 2]. In recent years, wavelets have been established many different fields of science and engineering. Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [3, 4]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations, a detailed survey on these papers can be found in [5 & 6]. Namely, Yousefi et al. [7] have introduced a new cosine and sine wavelet. Shiralashetti and Mundewadi [8] introduced Bernoulli wavelet method for solving Fredholm integral equations. In this paper, we introduced a new approach for solving integral equations using cosine and sine (CAS) wavelets.

2. Properties of Wavelets:

Wavelets: Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter '*a*' and the translation parameter '*b*' vary continuously, we have the following family of continuous wavelets [7];

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\quad a,b\in R,\quad a\neq 0 \qquad (2.1)$$

If we restrict the parameters *a* and *b* to discrete values as $a=a_0^{-k}, b=p\,b_0\,a_0^{-k}, a_0>1, b_0>0$ and *p*, and *k* positive integer, from Eq. (2.1) we have the following family of discrete wavelets:

$\psi_{k,p}(t)=|a|^{-\frac{1}{2}}\psi(a_0^k t-pb_0)$, where $\psi_{k,p}(t)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$, then $\psi_{k,p}(t)$ forms an orthonormal basis.

Cosine and Sine wavelets: Cosine and Sine wavelet $C_{n,m}(t)=C(k,n,m,t)$ have four arguments; $k=1,2,3,\ldots$, $n=1,2,3,\ldots,2^k$, $m=0,1,\ldots,M-1$ and t is the normalized time.

For any positive integer k , the cosine and sine wavelets family is defined in the interval [0, 1) as follows;

$$C_{n,m}(t)=\begin{cases}2^{k/2}CAS_m(2^k t-n), & for \quad \frac{n-1}{2^k}\le t<\frac{n}{2^k}\\ 0, & Otherwise\end{cases} \tag{2.2}$$

where $CAS_m(t)=\cos(2m\pi t)+\sin(2m\pi t)$.

Equivalently, by computational procedure for any positive integer k , the CAS wavelets family is defined as follows;

$$C_{i}(t)=\begin{cases}2^{k/2}CAS_m(2^k t-n), & for \quad \frac{n-1}{2^k}\le t<\frac{n}{2^k}\\ 0, & Otherwise\end{cases} \tag{2.3}$$

where $i=n+2^k m$. By varying the values of i with respect to the collocation points $t_j=\frac{j-0.5}{N}, j=1,2,\ldots,N$, we get the CAS wavelet matrix of order $N\times N$, where $N=2^k M$.

3. Cosine and Sine wavelets Method of solution for Integral Equations:

Fredholm Integral equations: Consider the Fredholm integral equations,

$$u(t)=f(t)+\int_0^1 k_1(t,s)u(s)\,ds, \tag{3.1}$$

where $f(t)\in L^2[0,1),\ k_1(t,s)\in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t)$, $u(t)$, and $k_1(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: Let us first approximate $f(t)=X^T\Psi(t)$ and $u(t)=Y^T\Psi(t)$ (3.2)

Let the function $f(t)\in L^2[0,1]$ may be expanded as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}x_{n,m}C_{n,m}(t), \tag{3.3}$$

where

$$x_{n,m}=(f(t),C_{n,m}(t)). \tag{3.4}$$

In (3.4), (. , .) denotes the inner product.

If the infinite series in (3.3) is truncated, then (3.3) can be rewritten as:

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}x_{n,m}C_{n,m}(t)=X^T\Psi(t), \tag{3.5}$$

where X and $\Psi(t)$ are $N\times 1$ matrices given by:

$$\begin{aligned}X&=[x_{10},x_{11},\ldots,x_{1,M-1},x_{20},\ldots,x_{2,M-1},\ldots,x_{2^{k-1},0},\ldots,x_{2^{k-1},M-1}]^T\\ &=[x_1,x_2,\ldots,x_{2^{k-1}M}]^T,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\Psi(t)&=[C_{10}(t),C_{11}(t),\ldots,C_{1,M-1}(t),C_{20}(t),\ldots,C_{2,M-1}(t),\ldots,C_{2^{k-1},0}(t),\ldots,C_{2^{k-1},M-1}(t)]^T\\ &=[C_1(t),C_2(t),\ldots,C_{2^{k-1}M}(t)]^T.\end{aligned} \tag{3.7}$$

STEP 2: Next, approximate the kernel function as: $k_1(t,s)\in L^2([0,1]\times[0,1])$

$$k_1(t,s)=\Psi^T(t)K_1\Psi(t), \tag{3.8}$$

where K_1 is $2^k M \times 2^k M$ matrix, with

$$[K_1]_{ij} = (C_i(t), (k_1(t,s), C_j(s))).$$

i.e., $[K_1] = [\Psi^T(t)]^{-1} \cdot [k_1(t,s)] \cdot [\Psi(s)]^{-1} \ \Psi^T(t)$, (3.9)

STEP 3: Substituting Eq. (3.2) and Eq. (3.8)in Eq. (3.1), we have:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_1\Psi(s)\Psi^T(s)Yds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_1\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_1Y),$$

Then we get a system of equations as,

$$(I - K_1)Y = X. \qquad (3.10)$$

By solving this system obtain the vector CAS wavelet coefficients '*Y*' & substituting in step 4.

STEP 4: $u(t) = Y^T \ \Psi(t)$, This is the required approximate solution of Eq.(3.1).

Volterra integral equations: Consider the Volterra integral equations with convolution but non-symmetrical kernel,

$$u(t) = f(t) + \int_0^t k_2(t,s)u(s)\,ds, \quad t \in [0,1] \qquad (3.11)$$

where $f(t) \in L^2[0,1),\ k_2(t,s) \in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t)$, and $k_2(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: The Eq. (3.11) can be rewritten in Fredholm integral equations, with a modified kernel $\tilde{k}_2(t,s)$ and solved in Fredholm form [9] as,

$$u(t) = f(t) + \int_0^t \tilde{k}_2(t,s)u(s)\,ds\,,\ \tilde{k}_2(t,s) = \begin{cases} k_2(t,s), & 0 \le s \le t \\ 0, & t \le s \le 1. \end{cases} \qquad (3.12)$$

STEP 2: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 3: Next, we approximate the kernel function as: $\tilde{k}_2(t,s) \in L^2([0,1]\times[0,1])$,

$$k_2(t,s) = \Psi^T(t) \cdot K_2 \cdot \Psi(s) \qquad (3.13)$$

where K_2 is $2^k M \times 2^k M$ matrix, with $(K_2)_{ij} = (C_i(t), (\tilde{k}_2(t,s), C_j(s)))$.

i.e., $K_2 = [\Psi^T(t)]^{-1} \cdot [k_2(t,s)] \cdot [\Psi(s)]^{-1}$, (3.14)

STEP 4: Substituting Eq. (3.2) and Eq. (3.13)in Eq. (3.12), we have:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_2\Psi(s)\Psi^T(s)Yds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_2\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2Y),$$

Then we get a system of equations as, $(I - K_2)Y = X.$ (3.15)

By solving this system, we obtain the vector CAS wavelet coefficients '*Y*' & substituting in step 5.

STEP 5: $u(t) = Y^T \ \Psi(t)$, This is the required approximate solution of Eq. (3.11).

4. Numerical Test Problems

Test Problem 4.1. Letus consider the Fredholm integral equation of the second kind [10],

$$u(t) = t + \int_0^1 k(t,s)u(s)\,ds\,,\ \ 0 \le t \le 1 \qquad (4.1)$$

which has the exact solution $u(t) = \sec(1)\sin(t)$. Where $f(t) = t$ and $k_1(t,s) = \begin{cases} t, & t \le s \\ s, & s \le t. \end{cases}$

We applied the CAS wavelet approach and solved Eq. (4.1), we get the CAS wavelet coefficients 'Y 'and substitute in $u(t) = Y^T\Psi(t)$, we obtain the approximate solution, it is compared with the exact solution in Table 1. Error analysis is compared with the existing method is shown in table 2.

Table 1: Numerical results of the Test Problem 4.1.

t	Exact	CAS Wavelet
0.0625	0.1156	0.1158
0.1875	0.3450	0.3456
0.3125	0.5690	0.5699
0.4375	0.7841	0.7854
0.5625	0.9870	0.9886
0.6875	1.1745	1.1764
0.8125	1.3437	1.3457
0.9375	1.4919	1.4941

Table 2: Maximum error analysis of the Test Problem 4.1.

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)	$E_{\max}$(CAS)
8	5.15e-02	8.67e-03	4.34e-03	2.18e-03
16	1.39e-02	1.21e-03	1.07e-03	5.52e-04
32	3.59e-03	2.77e-04	2.68e-04	1.38e-04
64	9.07e-04	6.77e-05	6.72e-05	3.47e-05
128	2.27e-04	1.68e-05	1.68e-05	8.70e-06

Test Problem 4.2. Next, consider the Volterra integral equation of the second kind [11],

$$u(t) = \sin(t) + \int_0^t (t-s)u(s)\,ds, \qquad 0 \le t \le 1 \qquad (4.2)$$

which has the exact solution $u(t) = \frac{1}{2}(\sin t + \sinh t)$. Where $f(t) = \sin(t)$ and $k_2(t,s) = (t-s)$ .We applied the CAS wavelet approach and solved Eq. (4.2), we get the CAS wavelet coefficients 'Y 'and substitute these coefficients in $u(t) = Y^T\,\Psi(t)$, we obtain the approximate solution, it is compared with the exact solution in table 3. Error analysis is compared with the existing method is shown in table 4.

Table 3: Numerical results of the Test Problem 4.2.

t	Exact	CAS
0.0625	0.0625	0.0625
0.1875	0.1875	0.1874
0.3125	0.3125	0.3123
0.4375	0.4376	0.4373
0.5625	0.5630	0.5626
0.6875	0.6888	0.6883
0.8125	0.8155	0.8149
0.9375	0.9435	0.9428

Table 4: Maximum error analysis of the Test Problem 4.2.

N	$E_{\max}$(HW)	$E_{\max}$(LW)	$E_{\max}$(BW)	$E_{\max}$(CAS)
8	2.42e-02	2.07e-03	3.18e-03	7.11e-03
16	6.52e-03	5.38e-04	5.24e-04	1.85e-04
32	1.69e-03	1.36e-04	1.32e-04	4.74e-05
64	4.32e-04	3.43e-05	3.40e-05	1.19e-05
128	1.09e-04	8.60e-06	8.58e-06	3.01e-06

6. Conclusion: In this paper, we proposed the wavelet based numerical method for the solution of integral equations. Using the Cosine and Sine wavelets, integral equations are reduced to the system of algebraic equations with unknown coefficients. Solving the system of equations, we obtain the unknown coefficients with the help of Matlab and then we approximate solution. Numerical results are compared with exact solutions as shown in tables. Error analysis shows the accuracy and effectiveness of the proposed scheme, which has been justified through the numerical test problems.

Acknowledgement: It is a pleasure to thank the University Grants Commission (UGC), Govt. of India for the financial support under UGC-SAP DRS-III for 2016-2021:F.510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016.

References:

1. Wazwaz A. M., Linear and nonlinear integral equations methods and applications, Springer, 2011.
2. Delves L. M. and Mohammed J. L., Computational Methods for Integral Equations, Cambridge University Press, Oxford, 1983.
3. Chui C. K., Wavelets: A mathematical tool for signal analysis, in: SIAM Monographs on Mathematical Modeling and Computation, SIAM, Philadelphia, Pennsylvania, 1997.
4. Beylkin G., Coifman R. and Rokhlin V., Fast wavelet transforms and numerical algorithms I. Commun. Pure Appl. Math., 44(1991)141–183.
5. Lepik Ü. and Tamme E., Application of the Haar wavelets for solution of linear integral equations, Ant. Turk–Dynam. Sys. Appl. Proce., (2005), 395–407.
6. Liang X. Z., Liu M. and Che X., Solving second kind integral equations by Galerkin methods with continuous orthogonal wavelets, J. Comp. App. Math., 136(2001)149-161.
7. Yousefi S. and Banifatemi A., Numerical solution of Fredholm integral equations by using CAS wavelets, App. Math. Comp., 183, (2006), 458-463.
8. Shiralashetti S. C., Mundewadi R. A., Bernoulli wavelet based numerical method for solving fredholm integral equations of the second kind, J. Inf. & Com. Sciences, 11(2), (2016), 111-119.
9. Lepik Ü., Numerical solution of evolution equations by the Haar wavelet method, Appl. Math. Comput. 185, (2007), 695–704.
10. Maleknejad K. and Mirzaee F., Using rationalized haar wavelet for solving linear integral equations, App. Math. Comp., 160, (2005), 579 – 587.
11. Blyth W. F., May R. L. and Widyaningsih P., Volterra integral equations solved in fredholm form using walsh functions, ANZIAM J., 45, 2004, C269-C282.

***