DYNAMIC OF GENERALIZED N - TOPOLOGY

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Abstract: In this paper, we introduce the structure of generalized N-topological spaces. The notions of $N\mu$-open sets, $N\mu$-closed sets, $N\mu$-interior, $N\mu$-closure and N^*g-continuous are also introduced and several characterizations of them are obtained.

Keywords: Generalized N-Topology, $N\mu$-Closed Sets, $N\mu$-Closure, $N\mu$-Interior, $N\mu$-Open Sets, N^*g-Continuous.

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1. Introduction: In 1963, Levine[4] introduced semi-open sets and semi continuity in topological spaces. After him, many researchers introduced similar weaker forms of open sets such as α-open sets, feebly open sets, pre-open sets and β-open sets. It was Csaszar[1], who observed the common features in all these open sets and brought all these open sets under one umbrella by defining γ-open sets.

Let X be a non-empty set. Let $\Gamma(X)$ be the collection of all mappings $\gamma: P(X) \to P(X)$ possessing the property of monotonicity. A subset A of X is said to be γ-open if $A \subseteq \gamma(A)$. The collection μ of all γ-open sets contains $\emptyset$ and is closed under arbitrary union. But it need not contain X and need not be closed under finite intersection. Such a collection is given the nomenclature, generalized topology.

In 2016, Thivagar et al.[2] introduced the structure of N-topology which is a non-empty set equipped with N-arbitrary topologies. In this paper, we have introduced generalized N-topological spaces.

2. Preliminaries: In this section, we discuss some basic definitions which will be useful for this paper.
Definition 2.1 [5]: A non-empty family μ of subsets of a non-empty set X is called a generalized topology, if $\emptyset\in\mu$ and arbitrary union of members of μ is again in μ. The pair (X,μ) is called a generalized topological space or GTS.

Definition 2.2 [2]: A quasi-pseudo metric on a non-empty set X is a function $d_1 : X \times X \to R^+ \cup \{0\}$ such that (i) $d_1(x,x) = 0$ for all $x \in X$.
(ii) $d_1(x,z) \le d_1(x,y)+d_1(y,z)$ for all $x,y,z\in X$, where R^+ is the set of all positive real numbers.

Definition 2.3 [2]: Let d_1 be a quasi-pseudo-metric on X, and let a function $d_2 : X \times X \to R^+ \cup \{0\}$ be defined by $d_2(x,y) = d_1(x,y)$ for all $x,y\in X$. Trivially d_2 is a quasi-pseudo-metric defined on X and we say that d_1 and d_2 are conjugate one another.

If d_1 is a quasi-pseudo-metric on X, then $B_{d_1}(x,k_1) = \{y : d_1(x,y) < k_1\}$, the open d_1-sphere with centre x and radius $k_1 > 0$. Classically, the collection of all d_1 spheres forms a basis for a topology, the obtained topology be denoted by τ_1 and is called the quasi-pseudo-metric topology of d_1. Similarly we get a topology τ_2 for X, due to the quasi-pseudo-metric d_2.

Definition 2.4 [2]: A non-empty set X equipped with two arbitrary topologies τ_1 and τ_2 is called a bitopological space and is denoted by (X,τ_1,τ_2).

Definition 2.5 [2]: Let d_1 and d_2 be conjugate, quasi-pseudo-metrics on X and define a function $d_3 : X \times X \rightarrow R^+ \cup \{0\}$ by

$$d_3(x,y) = \frac{[2d_1(y,x) + d_2(y,x)]}{3}, \forall x, y \in X$$

Then

(i) $d_3(x,z) = \frac{[2d_1(z,x) + d_2(z,x)]}{3} = 0$

for all $x \in X$

(ii) $d_3(x,z) = \frac{[2d_1(z,x) + d_2(z,x)]}{3}$

$$\leq \frac{[2(d_1(z,y) + d_1(y,x)) + (d_2(z,y) + d_2(y,x))]}{3} = d_3(x,y) + d_3(y,z) \text{ for all } x, y, z \in X .$$

Therefore, d_3 is a quasi-pseudo-metric on X which is called a Mean Conjugate(simply write M.C) of d_1,d_2 and d_1. For each $i= 1,2,3$, the quasi pseudo-metric d_i gives a topology τ_i whose base is $\{B_{d_i}(x,k_i)\}$, where $\{B_{d_i}(x,k_i) = \{y : d_i(x,y) < k_i\}$. Thus we define a non-empty set X equipped with three arbitrary topologies τ_1,τ_2, and τ_3 is called a tritopological space and is denoted by $(X,3\tau)$ or (X,τ_1,τ_2,τ_3). Generally, let $d_1,d_2,\cdots d_{N-1}$ be quasi-pseudo-metrics on X, d_1 and d_2 be conjugate and $d_3,d_4,\cdots d_{N-1}$ be M.C of d_1,d_2 and d_1; d_1,d_2,d_3 and d_1;$\cdots$;$d_1,d_2,\cdots ,d_{N-2}$ and d_1, respectively. Define a function $d_N : X \times X \rightarrow R^+ \cup \{0\}$ by

$$d_N(x,y) = \frac{\left[d_1(y,x) + \sum_{i=1}^{N-1} d_i(y,x) \right]}{N}$$

$\forall x, y \in X$. We can easily verify that d_N is a quasi-pseudo-metric on X. Also we note that for each N, $d_N(x,y) \neq d_N(y,x)$, for all $x,y \in X$ and d_N is called a Mean Conjugate(simply write M.C) of d_1, $d_2,\cdots$, d_{N-1} and d_1. For each $i=1,2,\cdots,N$, the quasi-pseudo metric d_i gives a topology τ_i whose basis is $\{B_{d_i}(x,k_i)\}$, where $B_{d_i}(x,k_i) = \{y : d_i(x,y) < k_i\}$. Thus we define a non-empty set equipped with N-arbitrary topologies $\tau_1,\tau_2,\cdots ,\tau_N$ is called a N-topological space and is denoted by $(X,N\tau)$ or$(X,\tau_1,\tau_2,\cdots ,\tau_N)$.

Definition 2.6 [2]: Let X be a non-empty set, $\tau_1,\tau_2,\ldots,\tau_N$ be N-arbitrary topologies on X and let the collection $N\tau$ be defined by $N\tau = \{S \subseteq X : S = (\cup_{i=1}^N A_i) \cup (\cap_{i=1}^N B_i) \, , A_i, B_i \in \tau_i\}$ satisfying the following axioms:

(i) $X,\emptyset \in N\tau$

(ii) $\cup_{i=1}^{\infty} S_i \in N\tau \forall S_i \in N\tau$

(iii) $\cap_{i=1}^{n} S_i \in N\tau \forall S_i \in N\tau$

The pair $(X,N\tau)$ is called a N-topological space.

3. Generalized N **– Topology:** In this section, we introduce the notion of generalized N-topological spaces.

Definition 3.1 Let X be a non-empty set. Let $\mu_1,\mu_2,\ldots,\mu_N$ be N arbitrary generalized topologies defined on X and the collection N_μ be defined by $N_\mu = \{C \subseteq X : C = (\cup_{i=1}^N A_i) \cup (\cap_{i=1}^N B_i) \, , A_i, B_i \in \mu_i\}$ satisfying the following axioms:

(i) $\emptyset \in N\mu$

(ii) $\cup_{i=1}^{\infty} C_i \in N\mu \forall C_i \in N\mu$

The pair $(X,N\mu)$ is called a generalized N-topological space and the elements in the collection $N\mu$ are called $N\mu$- open sets on X. A subset A of X is said to be $N\mu$- closed if its complement is $N\mu$ -open. The set of all $N\mu$ -open sets and $N\mu$ -closed sets are, respectively, denoted by $N\mu O(X)$ and $N\mu C(X)$.

Example 3.2: Let $X = \{a,b,c,d,e\}, \mu_1 O(X) = \{\emptyset,\{a,b\},\{b,c\}\}, \mu_2 O(X) = \{\emptyset,\{d\},\{c,d\}\}$ and $\mu_3 O(X)=\{\emptyset,\{c\},\{a,c\}\}$. Then $3\mu O(X)=\{\emptyset,\{a\}, \{c\},\{d\},\{a,b\},\{a,c\},\{b,c\},\{c,d\},\{a,d\},\{a,b,c\}, \{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$. $(X,3\mu)$ is a generalized tri-topological space.

Theorem 3.3: Let $(N\mu)_1$ and $(N\mu)_2$ be two generalized N-topological spaces on X. Then $(N\mu)_1 \cap (N\mu)_2$ is also a generalized N-topology on X.

Proof:

1. $\emptyset \in (N\mu)_1 \cap (N\mu)_2$
2. Let $\{C_i\}_{i\in I} \in (N\mu)_1 \cap (N\mu)_2$. Then

$C_i \in (N\mu)_1$ and $C_i \in (N\mu)_2$ $\forall i \in I$ Therefore $\bigcup_{i\in I} C_i \in (N\mu)_1$ and $\bigcup_{i\in I} C_i \in (N\mu)_2$ and hence $\bigcup_{i\in I} C_i \in (N\mu)_1 \cap (N\mu)_2$

Thus intersection of two generalized
N-topologies is again a generalized N-topology.

Remark 3.4: Union of two generalized
N-topologies need not be a generalized N-topology.

Example 3.5: Let $X = \{a,b,c,d\}$, $\mu_1 O(X) = \{\emptyset,\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\mu_2 O(X) = \{\emptyset,\{c\},\{a,c\}\}$. Then $2\mu O(X) = \{\emptyset,\{c\},\{a,b\}, \{a,c\},\{b,c\},\{a,b,c\}\}$.
Now for the generalised topologies $\mu_1' O(X) = \{\emptyset,\{a\},\{d\},\{a,d\}\}$ and $\mu_2' O(X) = \{\emptyset,\{a, b\},\{b,c\},\{a,b,c\}\}$, we have $2\mu' O(X) = \{\emptyset, \{a\}, \{d\}, \{a,d\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X\}$. 2μ and $2\mu'$ are generalized bitopological spaces on X. But $2\mu \cup 2\mu' = \{\emptyset, \{a\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\}, X\}$ which is not a generalized bitopology on X, since $\{c\},\{d\} \in 2\mu$, but $\{c\} \cup \{d\} \notin 2\mu$.

Definition 3.6: The N-μ interior of a subset S of X denoted by $N\mu$-$int(S)$ is the union of all $N\mu$ open sets contained in S. The N-μ closure of S denoted by $N\mu$-$cl(S)$ is the intersection of all $N\mu$ closed sets containing S.

Theorem 3.7: Let $(X,N\mu)$ be a generalized N-topological space and let $A,B \subseteq X$. Then

1. $N\mu$-$int(A)$ is the largest $N\mu$ open set contained in A.
2. $N\mu$-$cl(S)$ is the smallest $N\mu$ closed set containing A.
3. $N\mu\text{-}int(\emptyset) = \emptyset$
4. $N\mu\text{-}cl(X) = X$
5. $A \subseteq B \Rightarrow N\mu\text{-}int(A) \subseteq N\mu\text{-}int(B)$
6. $A \subseteq B \Rightarrow N\mu\text{-}cl(A) \subseteq N\mu\text{-}cl(B)$
7. $N\mu\text{-}int(A\cup B) \supseteq N\mu\text{-}int(A) \cup N\mu\text{-}int(B)$
8. $N\mu\text{-}cl(A\cup B) \supseteq N\mu\text{-}int(A) \cup N\mu\text{-}int(B)$
9. $N\mu\text{-}int(A\cap B) \subseteq N\mu\text{-}int(A) \cap N\mu\text{-}int(B)$
10. $N\mu\text{-}cl(A\cap B) \subseteq N\mu\text{-}cl(A) \cap N\mu\text{-}int(B)$

Proof

1. By definition, $N\mu$-$int(A)$ is a $N\mu$ open set contained in A. Let W be a $N\mu$ open set contained in A. Then $W \subseteq \cup\{C$: C is an $N\mu$ open set contained in $A\} = N\mu\text{-}int(A)$. Therefore $N\mu$-$int(A)$ is the largest $N\mu$ open set contained in A.
2. Proof is similar to (i).
3. Proof is obvious.
4. Proof is obvious.
5. Let $A \subseteq B$. Then every $N\mu$ open set contained in A is also an $N\mu$ open set contained in B.
   Therefore $\cup\{C$: C is a $N\mu$ open set contained in $A\} \subseteq \cup\{D$: D is a $N\mu$ open set contained in $B\}$
   Hence $N\mu\text{-}int(A) \subseteq N\mu\text{-}int(B)$
6. Proof is similar to (v).
7. We know that $A \subseteq A\cup B$ and $B \subseteq A\cup B$

Therefore using (v)$N\mu\text{-}int(A) \subseteq N\mu\text{-}int(A \cup B)$.
Similarly $N\mu\text{-}int(B) \subseteq N\mu\text{-}int(A \cup B)$.
Therefore $N\mu\text{-}int(A) \cup N\mu\text{-}int(B) \subseteq N\mu\text{-}int(A \cup B)$.
8. Proof is similar to (vii).
9. We know that $A \supseteq A \cap B$ and $B \supseteq A \cap B$.
Therefore using (v) $N\mu\text{-}int(A) \supseteq N\mu\text{-}int(A \cap B)$.
Similarly $N\mu\text{-}int(B) \supseteq N\mu\text{-}int(A \cap B)$.
Therefore $N\mu\text{-}int(A) \cap N\mu\text{-}int(B) \supseteq N\mu\text{-}int(A \cap B)$.
10. Proof is similar to (ix).

Remark 3.8: Though in classical *N*-topology, equality hold for (viii) and (ix) of theorem 3.7, it need not hold in a generalised *N*- topological space.

Example 3.9: In example 3.2, let $A=\{a,b\}$ and $B=\{b,c\}$. Then $A \cap B=\{b\}$, $3\mu\text{–}int(A \cap B)=\emptyset$. But $3\mu\text{–}int(A)=\{a,b\}$, $3\mu\text{–}int(B)=\{b,c\}$ and hence $3\mu\text{–}int(A) \cap 3\mu\text{–}int(B)=\{b\}$. Thus equality doesn't hold for (viii) of theorem 3.7. Again if $C=\{c,d,e\}, D=\{a,d,e\}$. Then $3\mu\text{–}cl(C)=C$, $3\mu\text{–}cl(D)=D$ and hence $3\mu\text{–}cl(C) \cup 3\mu\text{–}cl(D)=\{a,c,d,e\}$. But $C \cup D = \{a,c,d,e\}$ and $3\mu\text{–}cl(C \cup D)=X$. Thus equality doesn't hold for (ix) of theorem 3.7.

Theorem 3.10: Let $(X, N\mu)$ be a generalized *N*-topological space and $A \subseteq X$. Then
(i) $N\mu\text{-}int(A) = X\text{–}N\mu\text{-}cl(X\text{–}A)$.
(ii) $N\mu\text{-}cl(A) = X\text{–}N\mu\text{-}int(X\text{–}A)$.
Proof:
1. Let $x \in N\mu\text{–}int(A)$. Then $x \in G$ for some $N\mu$-open set G contained in A. That is $x \notin X\text{–}G$, where $X\text{–}G$ is a $N\mu$ closed set containing $X\text{–}A$.
Therefore $x \notin N\mu cl(X\text{–}A)$ which implies $x \in X\text{–}N\mu cl(X\text{–}A)$.
Similarly, if $x \in X\text{–}N\mu cl(X\text{–}A)$ then $x \notin N\mu cl(X\text{–}A)$. Hence $\exists$ a $N\mu$ closed set F containing $X\text{–}A$ such that $x \notin F$. Thus $x \in X\text{–}F$ which is a $N\mu$ open set contained in A. Hence $x \in N\mu int(A)$.
2. $x \in N\mu cl(A) \Leftrightarrow x \in F\ \forall\ N\mu$ closed set $F \subseteq A \Leftrightarrow x \notin X\text{–}F\ \forall\ N\mu$ open set $X\text{–}F \supseteq X\text{–}A \Leftrightarrow x \notin N\mu int(X\text{–}A)$ $\Leftrightarrow x \in X\text{–}N\mu int(A)$.

Theorem 3.11: Let $(X,N\mu)$ be a generalized *N*-topological space and $A \subseteq X$. Then
1. $N\mu\text{-}int(A) \supseteq \mu_1 int(A) \cup \mu_2 int(A) \cup \cdots \cup \mu_N int(A)$.
2. $N\mu\text{-}cl(A) \subseteq \mu_1 cl(A) \cap \mu_2 cl(A) \cap \cdots \cap \mu_N cl(A)$.

Proof:
1. Let $x \in \mu_1 int(A) \cup \mu_2 int(A) \cup \cdots \cup \mu_N int(A)$. Then $x \in \mu_i int(A)$ for some i. So, there exists a μ_i open set G containing x such that $G \subseteq A$. But every μ_i open set is also a $N\mu$ open set $\forall i$. Hence G is a $N\mu$ open set containing x such that $G \subseteq A$. Therefore $x \in N\mu\text{-}int(A)$. Hence $N\mu\text{-}int(A) \supseteq \mu_1 int(A) \cup \mu_2 int(A) \cup \cdots \cup \mu_N int(A)$.
2. Since (i) is true for every subset A of X replacing A by $X\text{–}A$ we get, $N\mu\text{-}int(X\text{–}A) \supseteq \mu_1 int(X\text{–}A) \cup \mu_2 int(X\text{–}A) \cup \cdots \cup \mu_N int(X\text{–}A)$.

Taking complements on both sides and applying demorgan's law and theorem 3.10, we get the desired result.

Remark 3.12: Equality need not hold in theorem 3.11.

Example 3.13: In example 3.2, let $A=\{a,d\}$. Then $\mu_1 int(A)=\emptyset$, $\mu_2 int(A)=\{d\}$, $\mu_3 int(A)=\emptyset$ and hence $\mu_1 int(A) \cup \mu_2 int(A) \cup \mu_3 int(A)=\{d\}$. But $3\mu int(A) = \{a,d\}$. Thus equality doesn't hold for (i) of theorem 3.11. Again if $B = \{b,c,e\}$, then $\mu_1 cl(B) = X, \mu_2 cl(B) = \{a,b,c,e\}$, $\mu_3 cl(B)=X$ and hence $\mu_1 cl(B) \cap \mu_2 cl(B) \cap \mu_3 cl(B) = \{a,b,c,e\}$. But $3\mu cl(B) = \{b,c,e\}$.Hence equality doesn't hold for (ii) of theorem 3.11.

Definition 3.14: Let $f : (X,N\mu) \rightarrow (Y,N\nu)$ be a function where X and Y are two generalized *N*-topological spaces. f is called N^*g-continuous if for every $N\nu$-open set U in Y, $f^{-1}(U)$ is a $N\mu$-open set in X.

Theorem 3.15: Let $f : (X,N\mu) \rightarrow (Y,N\nu)$ be a function where X and Y are two N-generalized topological spaces. Then the following are equivalent.

1. f is N^*g-continuous.
2. For every $N\nu$-closed set F in Y, $f^{-1}(F)$ is a $N\mu$-closed set in X.
3. For every subset A of X, $f(N\mu\text{-}cl(A)) \subseteq N\nu\text{-}cl(f(A))$.
4. For every subset B of Y, $N\mu\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(N\nu\text{-}cl(B))$.
5. For every subset B of Y, $f^{-1}(N\nu\text{-}int(B)) \subseteq N\mu\text{-}int(f^{-1}(B))$.

Proof (i)⇒(ii) Let f be N^*g-continuous. Then by definition, $f^{-1}(U)$ is a $N\mu$-open set in X,for every $N\nu$-open set U in Y . Let F be a $N\nu$-closed set in Y. Then F^c is an $N\nu$-open set in Y. Hence $f^{-1}(F^c)$ is $N\mu$-open in X. But $f^{-1}(F^c) = (f^{-1}(F))^c$. Therefore $(f^{-1}(F))^c$is $N\mu$-open in X. So $f^{-1}(F)$ is $N\mu$-closed in X.

(ii)⇒(iii) Let us assume that for every $N\nu$-closed set F in Y, $f^{-1}(F)$ is a $N\mu$-closed set in X. Let A be a subset of X. Now $N\nu\text{–}cl(f(A))$ is a $N\nu$-closed subset of Y. Hence by assumption, $f^{-1}(N\nu\text{-}cl(f(A)))$ is a $N\mu$-closed subset of X. Also it contains A. But $N\mu\text{-}cl(A)$ is the smallest $N\mu$-closed set containing A. Therefore $N\mu\text{-}cl(A) \subseteq f^{-1}(N\nu\text{-}cl(f(A)))$. Hence $f(N\mu\text{-}cl(A))\subseteq N\nu\text{-}cl(f(A))$.

(iii)⇒(iv) Let us assume that for every subset A of X,$f(N\mu\text{-}cl(A)) \subseteq N\nu\text{-}cl(f(A))$. Let B be a subset of Y . Then $f^{-1}(B)$ is a subset of X. Replacing A by $f^{-1}(B)$ in (iii), we get $f(N\mu\text{-}cl(f^{-1}(B))) \subseteq N\nu\text{-}cl(B)$. Hence $N\mu\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(N\nu\text{-}cl(B))$.

(iv)⇒(v) Let B be a subset of Y . Assume (iv) is true. Replacing B by B^c in (iv), we get, $N\mu\text{-}cl(f^{-1}(B^c))\subseteq f^{-1}(N\nu\text{-}cl(B^c))$. Taking complement on both side we get, $X\text{–}N\mu\text{-}cl(f^{-1}(B^c))\supseteq X\text{–}f^{-1}(N\nu\text{-}cl(B^c))$ which implies $X\text{–}N\mu\text{-}cl(f^{-1}(B)^c)\supseteq f^{-1}(Y\text{–}N\nu\text{-}cl(B^c))$. Using theorem 3.10, we get $N\mu\text{-}int(f^{-1}(B)) \supseteq f^{-1}(N\nu\text{-}int(B))$.

(v)⇒(i) Assume (v) is true. Let U be a $N\nu$-open set in Y . Using (v), we get $f^{-1}(N\nu\text{-}int(U))\subseteq N\mu\text{-}int(f^{-1}(U))$. Since U is $N\nu$-open, $f^{-1}(U)\subseteq N\mu\text{-}int(f^{-1}(U))$. But always $f^{-1}(U)\supseteq N\mu\text{-}int(f^{-1}(U))$. Therefore we get $f^{-1}(U)=N\mu\text{-}int(f^{-1}(U))$. Hence $f^{-1}(U)$ is $N\mu$-open in X. Therefore f is N^*g-continuous.

4. Conclusion: In this paper, we have introduced a new structure of generalized N-topology on a nonempty set. We have defined $N\mu$-interior, $N\mu$-closure and discussed some of their properties. We have also defined N^*g-continuous functions between generalized-N-topological spaces and established its characterizations. In future, this study can be extended to apply other concepts of topology in generalized N-topology.

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