CONTROLLING OF TRUNCATION ERRORS IN NUMERICAL SOLUTION OF HEAT TRANSFER PROBLEMS

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Abstract: Heat transfer problems often require extensive parametric studies to understand the influence of some variables on the solution in order to choose the right set of variables and to answer some "what-if" questions. This is an iterative process that is extremely tedious and time-consuming if done by hand. Computers and numerical methods are ideally suited for such calculations, and a wide range of related problems can be solved by minor modifications in the code or input variables. Today it is almost unthinkable to perform any significant optimization studies in engineering without the power and flexibility of computers and numerical methods.

Keywords: Error, Discretization, Numerical, Heat transfer, Grid.

Introduction: Numerical solution of heat transfer is a broad term which denotes the procedures for the solution. The objective in any heat transfer calculation is the determination of the rate of heat transfer to or from some surface or object. In conduction problems, this requires finding the temperature gradient in the material at its surface. In convection problems, the temperature gradient is required. [1]. In a fluid flowing over a surface is needed to find the heat flux at that surface. In both cases, the determination of the complete temperature distribution in the region of interest is needed as a first step. In convection one must also find the velocity distribution. Thus, a full solution of the energy equation and perhaps also the equations of motion is required. These are partial differential equations, possibly coupled. There are two methods to solve the problems related to heat transfer.

In the first, the equations are simplified for example, by linearization, or by neglecting terms considered sufficiently small, or by the assumption of constant properties, or by some other technique until an equation or system of equations is obtained for which an analytical solution can be found. It could be said that an exact analytical solution will be obtained for an approximate problem. The solution will, to some extent, be in error, and it will not normally be possible to estimate the magnitude of this error without recourse to external information such as an experimental result.

The second approach is to use a numerical method. [2]. In this, the continuous solution region is, replaced by a net or grid of lines and elements. The solution variables temperature, velocity, etc. are not obtained at all of the infinite number of points in the solution region, but only at the finite number of nodes of the grid, or at points within the finite number of elements.[3]. The differential equations are replaced by set of linear (or, rarely, nonlinear) algebraic equations, which must and can be solved on a computer.

Research Methodology: This is a review paper based on the numerical solution techniques of heat transfer. Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very use full, especially when the geometry of the object of interest is irregular, of the boundary conditions are nonlinear. In heat transfer analysis, some bodies are considered as a 'lump'. In a 'lump' interior temperature remains constant during heat transfer. The temperature of such bodies can be taken as a function of time only. Freezing of food, cooking of food, boiling of eggs are some examples of heat transfer problems in daily life. The growth rate of microorganism in a food product in environmental temperature is another example of heat transfer. [4]. In numerical analysis, three different approaches are commonly used; the finite difference, the finite volume and the finite element methods.

Discretization: Discretization is a cornerstone of numerical techniques i.e. numerical solution. An analytical solution to a partial differential equation gives us the value of f as a function of the independent variables (x_y,z,t). On the other hand, the numerical solution provides us the value of f at a *discrete* number of points in the domain. These points are called *grid points*, or sometimes as *nodes* or *cell centroids*, depending on the method. The process of converting our governing transport equation into a set of equations for the discrete values of f is called the *discretization process* and the specific methods employed to bring about this conversion are called *discretization method* .

The conversion of a differential equation into a set of discrete algebraic equations requires the discretization of space. This is accomplished by means of mesh generation. Mesh generation divides the domain of our interest into elements or cells, and

associates with each element or cell one or more discrete values of $\boldsymbol{f}$.

Since our aim is to get an answer to the original differential equation, it is appropriate to check whether our algebraic equation set really gives us this. When the number of grid points is small, the departure of the discrete solution from the exact solution is expected to be large. A well-behaved numerical scheme will tend to the exact solution as the number of grid points is increased. The rate at which it tends to the exact solution depends on the type of profile assumptions made in obtaining the discretization. No matter what discretization method is employed, all well-behaved discretization methods should tend to the exact solution when a large enough number of grid points is employed.

Discretization Error: The discretization error involved in numerical methods is due to replacing the derivatives by differences in each step, or the actual temperature distribution between two adjacent nodes by a straight line segment.

Consider the variation of the solution of a transient heat transfer problem with time at a specified nodal point [6]. Both the numerical and actual (exact) solutions coincide at the beginning of the first time step, as expected. But the numerical solution deviates from the exact solution as the time t increases. The difference between the two solutions at $t = \Delta t$ is due to the approximation at the first time step only and is called the local discretization error. We can expect the situation to get worse with each step since the second step uses the erroneous result of the first step as its starting point and adds a second local discretization error on top of it. The accumulation of the local discretization errors continues with the increasing number of time steps, and the total discretization error at any step is called the global or accumulated discretization error. Taylor series expansion represent the error involved in the finite difference approximation. Taylor series expansion of the temperature at a specified nodal point m about time t_i is given as

$$T\left(X_{m,}t_{i,} + \Delta t\right) = T\left(X_{m,}t_{i,}\right) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 T(x_m,t)}{\partial t^2} + \cdots \quad (1)$$

For a sufficiently small time step, these terms decay rapidly as the order of derivative increases, and their contributions become smaller and smaller. The local discretization error is the formulation error associated with a single step and gives an idea about the accuracy of the method used. However, the solution results obtained at every step except the first one involve the accumulated error up to that point, and the local error alone does not have much significance. Thus we conclude that the local discretization error is proportional to the square of the step size Δt^2, while the global discretization error is proportional to the step size Δt itself. Therefore smaller the mesh size smaller will be the error. [5]. Halving the step size will reduce the global discretization error by half. The discretizatin error approaches zero as the difference quantities such as Δx and Δt approach the differential quantities such as dx and dt.

Round-off error: If we had a computer that could retain an infinite number of digits for all numbers, the difference between the exact solution and the approximate (numerical) solution at any point would entirely be due to discretization error. But in every computer (or calculator) we represent numbers using a finite number of significant digits. The default value of the number of significant digits for many computers is 7. Which is referred to as single precision. But the user can perform the calculations using 15 significant digits for the numbers. Performing calculations in double precision will require more computer memory and a longer execution time. In single precision mode with seven significant digits, a computer registers the number 12345.666666 as 12345.67 or 12345.66. depending on the method of rounding the computer uses. In the first case. The excess digits are said to be rounded to the closest integer. Whereas in the second case they are said to be chopped off. Therefore, the numbers $a = 34567.12345$ and $b = 34567.12032$ are equivalent for a computer that performs calculations using seven significant digits. Such a computer would give $a - b = 0$ instead of the true value 0.00313.

The error due to retaining a limited number of digits during calculations is called the round-off error. This error is random in nature and there is no easy and systematic way of predicting it.

Round-off error depends on

1. The number of calculations.
2. The method of rounding off.
3. The type of computer.
4. The sequence of calculations.

Controlling the Error in Numerical Methods: The total error in any result obtained by a numerical method is the sum of the *discretization error*, which decreases with decreasing step size, and the *round-off error*. Which increases with decreasing step size. Therefore, decreasing the step size too much in order to get more accurate results may actually backfire and give less accurate results because of a faster increase in the round-off error. We should be careful not to let round-off error get out of control by avoiding a large number of computations with very small numbers.

In practice, we do not know the exact solution of the problem, and thus we cannot determine the magnitude of the error involved in the numerical method. Knowing that the global discretization error is proportional to the step size is not much help either since there is no easy way of determining the

value of the proportionally constant. Besides, the global discretization error alone is meaningless without a true estimate of the round-off error. Therefore, we recommend the following practical procedures to assess the accuracy of the results obtained by a numerical method.

Start the calculations with a reasonable mesh size Δx (and time step size Δt for transient problems) based on experience. Then repeat the calculations using a mesh size of $\frac{\Delta x}{2}$. If the results obtained by halving the mesh size do not differ significantly from the results obtained with the full mesh size, we conclude that the discretization error is at an acceptable level. But if the difference is larger than we can accept, then we have to repeat the calculations using a mesh size $\frac{\Delta x}{4}$ or even a smaller one at regions of high temperature gradients. We continue in this manner until halving the mesh size does not cause discretization error is reduced to an acceptable level.

Repeat the calculations using double precision holding the mesh size constant. If the changes are not significant, we conclude that the round-off error is not a problem. But if the changes are too large to accept, then we may try to reduce the total number of calculations by increasing the mesh size or changing the order of computations. But if the increased mesh size gives unacceptable discretization errors, then we may have to find a reasonable compromise.

It should always be kept in mind that the results obtained by any numerical method may not reflect any trouble spots in certain problems that require special consideration such as hot spots or areas of high temperature gradients. The results that seem quite reasonable overall may be in considerable error at certain locations. This is another reason for always repeating the calculations at least twice with different mesh sizes before accepting them as the solution of the problem. Most commercial software packages have built-in routines that vary the mesh size as necessary to obtain highly accurate solutions.

Consider a function T(x) and its derivatives to be single-valued, finite, and continuous with respect to x. The Taylor series expansion of T(x + Δ(x)) about T(x) may be written as

$$T(x+\Delta x) = T(x) + \left.\frac{dT}{dx}\right|_x \Delta x + \frac{1}{2!}\left.\frac{d^2T}{dx^2}\right|_x (\Delta x)^2 + \frac{1}{3!}\left.\frac{d^3T}{dx^3}\right|_x (\Delta x)^3 + \ldots \quad (2)$$

$$T(x-\Delta x) = T(x) - \left.\frac{dT}{dx}\right|_x \Delta x + \frac{1}{2!}\left.\frac{d^2T}{dx^2}\right|_x (\Delta x)^2 - \frac{1}{3!}\left.\frac{d^3T}{dx^3}\right|_x (\Delta x)^3 + \ldots \quad (3)$$

Adding Eqs. (2) and (3)

$$T(x+\Delta x) + T(x-\Delta x = 2T(x) + \left.\frac{d^2T}{dx^2}\right|_x (\Delta x)^2 + O\{(\Delta x)^4 \quad (4)$$

Where the last term in Eq. (4) represents terms in the fourth and higher powers of Δx. Equation (4) may be written as

$$\left.\frac{d^2T}{dx^2}\right|_x = \frac{T(x+\Delta x) + T(x-\Delta x) - 2T_{(x)}}{(\Delta x)^2} + O[(\Delta)^2] \quad (5)$$

The second order derivative of $T(x)$ in finite difference method is given by

$$\left.\frac{d^2T}{dx^2}\right|_{x_i} \approx \frac{T_{i+1} + T_{i-1} - 2T_i}{(\Delta x)^2} \quad (6)$$

Comparing Eq. (5) to Eq.(6), the finite difference approximation of $\frac{d^2T}{dx^2}$ has a truncation error of the order of magnitude of $(\Delta x)^2$.

If we subtract Eq. (3) from Eq. (2), and rearrange, We obtain

$$\left.\frac{dT}{dx}\right|_x \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} + O\{(\Delta x)^2 \quad (7)$$

The first order derivative of $T(x)$ in finite difference method is given by

$$\left.\frac{dT}{dx}\right|_{x_i} \approx \frac{T_{i+1} + T_{i-1}}{2\Delta x} \quad (8)$$

Comparing Eq. (7) to Eq. (8), the finite difference approximation of $\left.\frac{dT}{dx}\right|$ has a truncation error of the order of magnitude of $(\Delta x)^2$.

The other two approximations for the first derivative of T with respect to x have truncation errors of the order of magnitude of Δx. Thus, the central difference formulation is more precise than the other two.

Conclusion: The truncation errors are inherent in the finite difference method and cannot be eliminated. Truncation error refers to the error in a method, which occurs because some series (finite or infinite) is truncated to a fewer number of terms. Such errors are essentially algorithmic errors and we can predict the extent of the error that will occur in the method. The errors may be reduced by selecting a finer grid. In other words, smaller increments for space and time will reduce the truncation errors.

Numerical solutions are carried out to a finite number of significant figures; the numbers are rounded-off and thus, round-off errors are introduced. Round-off errors compound, and this may result in a large cumulative error. It is difficult to estimate the order of magnitude of the cumulative round-off errors. The use of smaller increments in space and time increases the accumulation of round-off errors, even though they lead to less truncation errors.

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